Math 319, Fall 2013, Term Test I

Techniques in Ordinary Differential Equations

Date: Friday, October 4

Lecture Section: 001

Name (printed): ____________________

UW Student ID Number: ____________________

Discussion Section: (circle)

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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

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Total	/25

1. Definitions and Theorems:

[1] (a) Consider a first-order differential equation $y' = f(x,y)$. State a condition sufficient to guarantee the existence of a solution through the point (x_0, y_0).

$f(x,y)$ is continuous in a region around (x_0, y_0).

[1] (b) Consider a first-order differential equation $M(x,y)\ dx + N(x,y)\ dy = 0$. State a condition sufficient to guarantee the equation is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

[4] 2. True/False:

(a) Consider a first-order differential equation $y' = f(x,y)$. If $f(x,y)$ is continuous at every point (x,y) then every solution $y(x)$ is defined over the whole domain $x \in \mathbb{R}$. [True / False]

(b) The integrating factor for the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}y + \frac{1}{x^2}$ is $\mu(x) = \frac{1}{x}$. [True / False] $\Rightarrow y' - \frac{1}{x}y = \frac{1}{x^2} \Rightarrow \mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

(c) A Bernoulli differential equation of the form $y' + P(x)y = Q(x)y^n$ can always be turned into a first-order linear differential equation by the transformation $v = y^{1-n}$. [True / False]

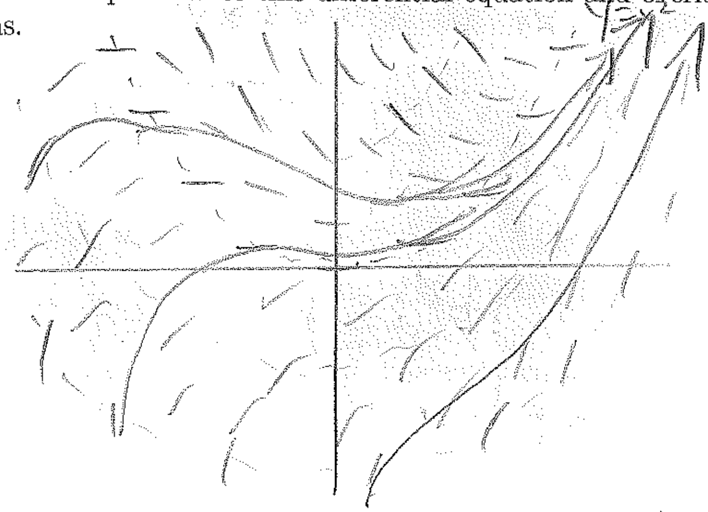
(d) Every differential equation of the form $M(x,y)\ dx + \cancel{N(x,y)}\ dy = 0$ can be made exact by some choice of integrating factor. [True / False]

3. Slope Fields:

Consider the first-order differential equation

$$\frac{dy}{dt} = -y + x^2. \qquad (1)$$

[2] (a) Sketch a slope field for this differential equation and overlay a few potential solutions.



$-y + x^2 = C$

$\Rightarrow y = x^2 - C$

$\frac{dy}{dx}$	$f(x)$
0	$y = x^2$
-1	$y = x^2 + 1$
1	$y = x^2 - 1$
-2	$y = x^2 + 2$
2	$y = x^2 - 2$

etc.

[2] (b) Show that $y(x) = x^2 - 2x + 2 + Ce^{-x}$ is a solution of (1) for all $C \in \mathbb{R}$.

$LHS = y' = 2x - 2 - Ce^{-x}$

$RHS = -y + x^2 = -(x^2 - 2x + 2 + Ce^{-x}) + x^2$

$= \cancel{-x^2} + 2x - 2 - Ce^{-x} \cancel{+x^2}$

$= 2x - 2 - Ce^{-x} = LHS$

$\Rightarrow$ Solution!

[2] (c) Find the particular solution of (1) for the initial condition $y(1) = 0$. What is the behavior of this solution in the limit $x \to \infty$?

$y(1) = 0 \Rightarrow 0 = (1)^2 - 2(1) + 2 + Ce^{-(1)}$

$\Rightarrow 0 = 1 + Ce^{-1}$

$\Rightarrow C = -e$

$\Rightarrow y(x) = x^2 - 2x + 2 - e^{-x+1}$

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} (\underbrace{x^2 - 2x + 2}_{\infty} - \underbrace{e^{-x+1}}_{0}) = \infty.$$

4. General Solutions:

Solve the following differential equations, including initial conditions if specified:

[3] (a) $2xy\frac{dy}{dx} = x\sqrt{x^2+y^2} + 2y^2$, $x>0$ Power homogeneous!

$\Rightarrow v = \frac{y}{x} \Rightarrow y = xv \Rightarrow y' = v + xv'$

$\Rightarrow 2x(xv)(v+xv') = x\sqrt{x^2+(xv)^2} + 2(xv)^2$

divide by x^2! $\Rightarrow 2v(v+xv') = \sqrt{1+v^2} + 2v^2$

$\Rightarrow 2vxv' = \sqrt{1+v^2}$

$\Rightarrow \int \frac{2v}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx$

$\Rightarrow 2\sqrt{1+v^2} = \ln(x) + C$

$\Rightarrow 1+v^2 = \left(\frac{\ln(x)+C}{2}\right)^2$

$\Rightarrow v = \pm\sqrt{\left(\frac{\ln(x)+C}{2}\right)^2 - 1}$

$\Rightarrow \frac{y}{x} = \pm\sqrt{\left(\frac{\ln(x)+C}{2}\right)^2 - 1}$

$\Rightarrow y(x) = \pm x\sqrt{\left(\frac{\ln(x)+C}{2}\right)^2 - 1}$

[4] (b) $\frac{dy}{dx} = \frac{x^2+y^2-y}{x(1-2y)}$, y(3)=0 (*Hint:* Recall standard exact form!)

$\Rightarrow \underbrace{-(x^2+y^2-y)}_{M}dx + \underbrace{x(1-2y)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = -2y+1 = \frac{\partial N}{\partial x} \Rightarrow$ exact!

$\Rightarrow \frac{\partial F}{\partial x} = -x^2-y^2+y \Rightarrow F(x,y) = -\frac{x^3}{3} - y^2x + yx + g(y)$

$\frac{\partial F}{\partial y} = x - 2xy$ $\qquad \frac{\partial F}{\partial y} = -2xy + x + g'(y)$

compare!

$g'(y) = 0 \Rightarrow g(y) = C$

$\Rightarrow -\frac{x^3}{3} - xy^2 + xy = C$ is solution.

$y(3)=0 \Rightarrow -\frac{(3)^3}{3} - (3)(0)^2 + (3)(0) = C \Rightarrow C = -9$

$\Rightarrow -\frac{x^3}{3} - xy^2 + xy = -9.$

5. Applications:

Consider a mixing tank with a total volume of 20 gallons, initially filled with 10 gallons of pure water. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water mixture) at a rate of 4 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallon per minute.

[1] (a) Use the given information to derive a differential equation which models the amount of salt in the tank. $V(t) = 10 + (4-2)t = 10 + 2t$

$$\Rightarrow \frac{dA}{dt} = (4)(0.5) - (2)\frac{A}{10+2t} \qquad A(0) = 0$$
$$= 2 - \left(\frac{2}{10+2t}\right)A$$

[3] (b) Find the general solution of the differential equation derived in part (a).

$$A' + \left(\frac{2}{10+2t}\right)A = 2 \qquad \mu(t) = e^{2\int \frac{1}{10+2t}dt} = e^{\ln(10+2t)} = 10+2t$$

$$\Rightarrow \frac{d}{dt}\left[(10+2t)A\right] = 2(10+2t)$$
$$\Rightarrow (10+2t)A = \frac{(10+2t)^2}{2} + C$$
$$\Rightarrow A(t) = 5 + t + \frac{C}{10+2t}$$

[2] (c) How much salt is in the tank when it is full? (Remember initial condition)

$$A(0) = 0 \Rightarrow 0 = 5 + \frac{C}{10} \Rightarrow C = -50$$

$$\Rightarrow \boxed{A(t) = 5 + t - \frac{50}{10+2t}}$$

Full $\Rightarrow V(t) = 10 + 2t = 20 \Rightarrow \boxed{t = 5}$

$$A(5) = 5 + (5) - \frac{50}{10+2(5)}$$
$$= 10 - \frac{50}{20} = 10 - 2.5 = 7.5 \text{ lbs}$$