Math 319, Fall 2013, Term Test I

Techniques in Ordinary Differential Equations

Date: Friday, October 4

Lecture Section: 002

Solutions

Name (printed): ______________________

UW Student ID Number: ______________________

Discussion Section: (circle)

Liu Liu:	**301**	**302**	**303**	**304**
Huanyu Wen:	**305**	**306**	**323**	**324**
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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
Page	Mark
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3	/6
4	/7
5	/6
Total	/25

1. Definitions and Theorems:

[1] (a) Consider a first-order differential equation $y' = f(x, y)$. State a condition sufficient to guarantee the uniqueness of a solution through the point (x_0, y_0).

$\frac{\partial f}{\partial y}$ is continuous in a region around (x_0, y_0)

[1] (b) Consider a (power) homogeneous differential equation $y' = F(y/x)$. State a substitution which will transform this into a separable differential equation.

$v = \frac{y}{x}$

[4] 2. True/False:

(a) The forward Euler method is the most accurate numerical scheme, as well as being the simplest. [True / False]

(b) An integrating factor for the nearly exact differential equation $y^2(x + 1)\, dx + 2xy\, dy = 0$ is $\mu(x) = e^x$. (Just check!) [True / False]

$M = y^2(x+1)e^x$, $N = 2xe^x y$ $\quad \frac{\partial M}{\partial y} = 2y(x+1)e^x$

$\frac{\partial N}{\partial x} = 2e^x y + 2xe^x y$ ✓

(c) A Bernoulli differential equation of the form $y' + P(x)y = Q(x)y^n$ can always be turned into a first-order linear differential equation by the transformation $v = y^{1-n}$. [True / False]

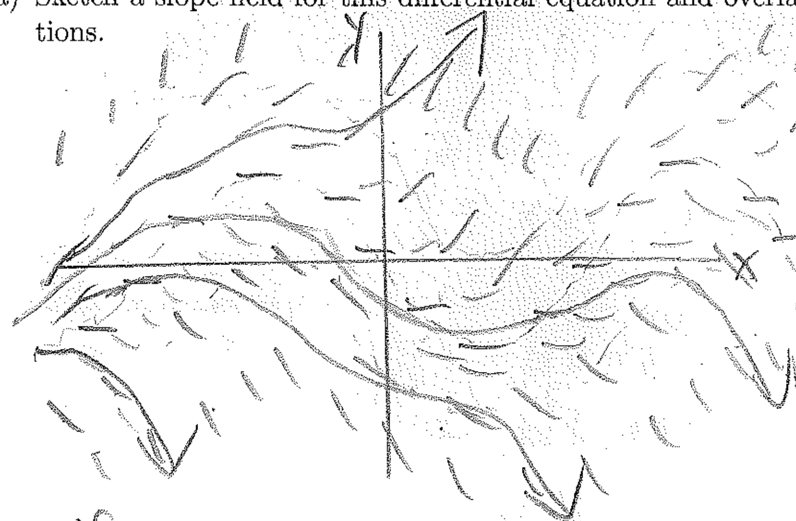
(d) For first-order linear differential equations of the form $y' + p(x)y = q(x)$, an integrating factor of the form $\mu(x) = e^{\int q(x)\, dx}$ is often needed. [True / False]

3. Slope Fields:

Consider the first-order differential equation

$$\frac{dy}{dt} = y + \sin(x). \qquad (1)$$

[2] (a) Sketch a slope field for this differential equation and overlay a few potential solutions.



$C = y + \sin(x)$

$y = -\sin(x)$

$\frac{dy}{dx}$	$f(x)$
0	$y = -\sin(x)$
1	$y = -\sin(x) + 1$
2	$y = -\sin(x) + 2$
−1	$y = -\sin(x) - 1$
−2	$y = -\sin(x) - 2$

[2] (b) ~~Show~~ Verify that $y(x) = -\frac{1}{2}(\sin(x) + \cos(x)) + Ce^x$ is a solution of (1) for all $C \in \mathbb{R}$.

$LHS = y' = -\frac{1}{2}(\cos(x) - \sin(x)) + Ce^x$

$= \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) + Ce^x$

$RHS = y + \sin(x) = \left[-\frac{1}{2}(\sin(x) + \cos(x)) + Ce^x\right] + \sin(x)$

$= \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) + Ce^x$

$= LHS$ ✓

$\Rightarrow$ Solution!

[2] (c) Find the particular solution of (1) for the initial condition $y(0) = -1/2$. What is the behavior of this solution in the limit $x \to \infty$?

$y(0) = \frac{1}{2} = -\frac{1}{2}(\sin(0) + \cos(0)) + Ce^{(0)}$

$\Rightarrow -\frac{1}{2} = -\frac{1}{2} + C \Rightarrow C = 0$

$\Rightarrow y(x) = -\frac{1}{2}(\sin(x) + \cos(x))$

Solution will oscillate forever.

4. General Solutions:

Solve the following differential equations, including initial conditions if specified:

[3] (a) $\dfrac{dy}{dx} = -\dfrac{2y^2+6xy-4}{3x^2+4xy+3y^2}$ (*Hint:* Recall standard exact form!)

$$\Rightarrow \underbrace{(2y^2+6xy-4)}_{M}dx + \underbrace{(3x^2+4xy+3y^2)}_{N}dy = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = 4y+6x = \frac{\partial N}{\partial x} \Rightarrow \text{exact!}$$

$$\Rightarrow \frac{\partial F}{\partial x} = 2y^2+6xy-4 \Rightarrow F(x,y) = 2xy^2+3x^2y-4x+g(y)$$

$$\frac{\partial F}{\partial y} = 3x^2+4xy+3y^2 \overset{\text{compare}}{\Leftrightarrow} \frac{\partial F}{\partial y} = 4xy+3x^2+g'(y)$$

$$\Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C \Rightarrow \boxed{2xy^2+3x^2y-4x+y^3 = C}$$

[4] (b) $x\dfrac{dy}{dx} + y - \cancel{x}y^2e^{2x} = 0,\ y(1) = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \cancel{x}e^{2x}y^2 \quad \text{Bernoulli! } n=2$$

$$v = y^{1-n} = y^{-1} \Rightarrow y = v^{-1} \Rightarrow y' = -v^{-2}(v')$$

$$\Rightarrow -v^{-2}v' + \frac{1}{x}v^{-1} = \cancel{x}e^{2x}v^{-2}$$

$$\Rightarrow v' - \frac{1}{x}v = -\cancel{x}e^{2x}$$

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{x}v\right] = -e^{2x}$$

$$\Rightarrow \frac{1}{x}v = -\int e^{2x}dx = -\frac{e^{2x}}{2} + C$$

$$\Rightarrow v = \frac{(C-e^{2x})x}{2}$$

$$\Rightarrow y^{-1} = \frac{(C-e^{2x})x}{2}$$

$$\Rightarrow y = \frac{2}{(C-e^{2x})x}$$

$$y(1) = 1 \Rightarrow 1 = \frac{2}{(C-e^2)}$$

$$\Rightarrow C - e^2 = 2$$

$$\Rightarrow C = 2 + e^2$$

$$\Rightarrow y(x) = \frac{2}{(2+e^2-e^{2x})x}$$

5. Applications:

Consider the reversible chemical reaction

$$X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y.$$

The differential equation governing the concentration of x can be given by

$$\frac{dx}{dt} = -\alpha x + \beta y \tag{2}$$

where $\alpha, \beta > 0$.

[2] (a) Use the conservation relation $x + y = C$ to rewrite (2) only in terms of the variable x (i.e. eliminate y).

$$y = C - x \Rightarrow \frac{dx}{dt} = -\alpha x + \beta(C - x)$$
$$= -\alpha x + \beta C - \beta x$$
$$= -(\alpha+\beta)x + \beta C$$

[3] (b) Solve the differential equation found in part (a) for $x(t)$. Use the conservation relationship to then solve for $y(t)$.

First-order linear!

$$\Rightarrow \frac{dx}{dt} + (\alpha+\beta)x = \beta C \qquad \mu(t) = e^{\int(\alpha+\beta)dt} = e^{(\alpha+\beta)t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{(\alpha+\beta)t} x\right] = \beta C e^{(\alpha+\beta)t}$$

$$\Rightarrow e^{(\alpha+\beta)t} x = \frac{\beta C}{\alpha+\beta} e^{(\alpha+\beta)t} + D$$

$$\Rightarrow x(t) = \frac{\beta C}{\alpha+\beta} + D e^{-(\alpha+\beta)t}$$

$$y(t) = C - x = C - \frac{\beta C}{\alpha+\beta} - D e^{-(\alpha+\beta)t}$$

[1] (c) What is the long-term behavior of the system? (i.e. What happens as $t \to \infty$?)

$$\lim_{t\to\infty} x(t) = \frac{\beta C}{\alpha+\beta} = \frac{1}{2} \quad (\text{for } \alpha = \beta = C = 1)$$

$$\lim_{t\to\infty} y(t) = C - \frac{\beta C}{\alpha+\beta} = \frac{1}{2}$$