Math 319, Fall 2013, Term Test II

Techniques in Ordinary Differential Equations

Date: Friday, November 15

Lecture Section: 001

SOLUTIONS

Name (printed): ______________________

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Discussion Section: (circle)

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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
Page	Mark
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4	/6
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1. Short Answer:

[1] (a) Consider the mechanical model $mx''(t) + cx'(t) + kx(t) = 0$. State a relationship on the constants m, c, and k such that the mechanism is *critically damped.*

$$mr^2 + cr + k = 0 \Rightarrow r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\boxed{\Rightarrow c^2 = 4mk}$$

[2] (b) Determine the Wronskian of $y_1(x) = \ln(x)$ and $y_2(x) = 1/x^2$. Can $y_1(x)$ and $y_2(x)$ constitute a fundamental solution set for a homogeneous second-order differerential equation (for $x > 0$)?

$$W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2$$

$$= \ln(x)\left(-\frac{2}{x^3}\right) - \frac{1}{x}\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{x^3}(-2\ln(x) - 1) \neq 0 \quad \text{for } x > 0$$

$\Rightarrow$ could be fundamental solution set!

[3] 2. True/False:

(a) The complementary solution of $y''(x) - 8y'(x) + 20y(x) = e^{4x}\sin(2x)$ is $y_c(x) = C_1 e^{4x}\sin(2x) + C_2 e^{4x}\cos(2x)$. [True / False]

$$\Rightarrow r^2 - 8r + 20 = 0 \Rightarrow r = \frac{8 \pm \sqrt{64 - 80}}{2} = 4 \pm 2i$$

(b) The amplitude (i.e. maximal value) of the particular solution of $y''(x) + 2y'(x) + 5y(x) = \cos(\omega x)$ does not depend on ω. [True / False]

resonance!

(c) Every point $x_0 \in \mathbb{R}$ is an ordinary point of $(x^2+1)y''(x) + xy'(x) - (x+1)y(x) = 0$. [True / False]

$P(x) = x^2 + 1 \neq 0$ ✓

3. Second-Order Differential Equations

[3] (a) Consider the following differential equation:

$$y''(x) + 4y'(x) + 5y(x) = g(x).$$

Determine the complementary solution $y_c(x)$ then set-up, but *do not attempt to evaluate*, the trial form $y_p(x)$ for the given choices of $g(x)$ below.

$\Rightarrow r^2+4r+5=0 \Rightarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$\Rightarrow y_c(x) = C_1 e^{-2x}\cos(x) + C_2 e^{-2x}\sin(x)$

(i) $g(x) = x^2 e^{-2x}$

$y_p(x) = Ax^2e^{-2x} + Bxe^{-2x} + Ce^{-2x}$

(ii) $g(x) = e^{-2x}\cos(x)$

$y_p(x) = Axe^{-2x}\cos(x) + Bxe^{-2x}\sin(x)$

[4] (b) Given that $y_1(x) = 1/x$ and $y_2(x) = 1/x^5$ are fundamental solutions of $x^2y''(x) + 7xy'(x) + 5y(x) = 0$, determine the particular solution of

$$x^2y''(x) + 7xy'(x) + 5y(x) = x.$$

[*Hint:* Remember the standard form for when applying variation of parameters!]

$W(y_1,y_2)(x) = y_1y_2' - y_1'y_2 = \left(\frac{1}{x}\right)\left(-\frac{5}{x^6}\right) - \left(-\frac{1}{x^2}\right)\left(\frac{1}{x^5}\right)$

$= -\frac{5}{x^7} + \frac{1}{x^7} = -\frac{4}{x^7}$

Note $g(x) = \frac{x}{x^2} = \frac{1}{x}$

$\Rightarrow u_1(x) = -\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx = -\int \frac{\left(\frac{1}{x^5}\right)\left(\frac{1}{x}\right)}{-\left(\frac{4}{x^7}\right)}dx$

$= \frac{1}{4}\int x\,dx = \frac{1}{8}x^2$

$u_2(x) = \int \frac{y_1(x)\,g(x)}{W(y_1,y_2)(x)}dx = \int \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)}{\left(-\frac{4}{x^7}\right)}dx$

$= -\frac{1}{4}\int x^5\,dx = -\frac{1}{24}x^6$

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = \left(\frac{1}{8}x^2\right)\left(\frac{1}{x}\right) + \left(-\frac{1}{24}x^6\right)\left(\frac{1}{x^5}\right)$

$= \frac{1}{8}x - \frac{1}{24}x = \frac{1}{12}x$

4. Power Series Solutions

[2] (a) Determine the radius of convergence of the power series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{(2n)!}(x-1)^n$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{n+1}}{[2(n+1)]!}\frac{(2n)!}{(-1)^n 3^n}\right| = \lim_{n\to\infty}\frac{3}{(2n+2)(2n+1)} = 0$$

$\Rightarrow$ Converges everywhere!

[2] (b) Write the following expression as a single summation whose common term is x^n:

$$\sum_{n=2}^{\infty}(n-1)n^2 a_n x^{n-2} + x\sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty}(n+1)(n+2)^2 a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n$$

$$= \sum_{n=0}^{\infty}\left[(n+1)(n+2)^2 a_{n+2} + n a_n\right]x^n$$

[2] (c) Suppose that a particular differential equation has a power series solution centered at $x_0 = 0$ generated by the recursion relation

$$a_{n+2} = \frac{a_{n+1} - (n+1)a_n}{n+2}, \quad n \geq 0.$$

Determine up to the x^3 term of the solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$n=0 \Rightarrow a_2 = \frac{a_1 - a_0}{2}$$

$$n=1 \Rightarrow a_3 = \frac{a_2 - 2a_1}{3} = \frac{\left(\frac{a_1}{2} - \frac{a_0}{2}\right) - 2a_1}{3} = -\frac{1}{6}a_0 - \frac{1}{2}a_1$$

$$\Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right)x^2 + \left(-\frac{a_0}{6} - \frac{a_1}{2}\right)x^3 + \cdots$$

$$= a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right) + a_1\left(x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots\right)$$

5. Laplace Transforms:

[2] (a) Evaluate the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2-4s+8}\right\}$$ no real factors

$$= \mathcal{L}^{-1}\left\{\frac{2}{(s^2-4s+4)+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= e^{2x}\sin 2x$$

[4] (b) Use the Laplace transform method to evaluate the following initial value problem:

$$\frac{d^2y}{dx^2} - y(x) = e^x, \quad y(0) = 0,\ y'(0) = 1/2.$$

Laplace transform $\Rightarrow$ $[s^2Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=\frac{1}{2}}] - Y(s) = \frac{1}{s-1}$

$$\Rightarrow (s^2-1)Y(s) = \frac{1}{s-1} + \frac{1}{2} = \frac{2+s-1}{2(s-1)} = \frac{s+1}{2(s-1)}$$

$$\Rightarrow Y(s) = \frac{s+1}{2(s-1)(s^2-1)} = \frac{s+1}{2(s-1)(s-1)(s+1)} = \frac{1}{2(s-1)^2}$$

$$\Rightarrow y(x) = \frac{1}{2}xe^x$$