Math 319, Fall 2013, Term Test II

Techniques in Ordinary Differential Equations

Date: Friday, November 15

Lecture Section: 002

SOLUTIONS

Name (printed): ____________________________

UW Student ID Number: ____________________________

Discussion Section: (circle)

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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
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1. Short Answer:

[1] (a) Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous second-order differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

satisfying $W(y_1, y_2)(x) \neq 0$. State the general solution form of the differential equation.

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

[2] (b) Determine the values a for which the following differential equations exhibit *oscillatory* behavior:

(i) $y''(x) + ay'(x) + y(x) = 0$

$r^2 + ar + 1 = 0$

$\Rightarrow r = \frac{-a \pm \sqrt{a^2-4}}{2} \Rightarrow a^2 < 4$

$\Rightarrow -2 < a < 2$

(ii) $ay''(x) - 2y'(x) + 2y(x) = 0$

$ar^2 - 2r + 2 = 0$

$\Rightarrow r = \frac{2 \pm \sqrt{4-8a}}{2} \Rightarrow 4 - 8a < 0$

$a > \frac{1}{2}$

[3] 2. True/False:

(a) The mechanical mechanism $x''(t) + cx'(t) + 4x(t) = 0$ is *critically damped* at the value $c = 4$. [True / False]

$r^2 + cr + 4 = 0$

$\Rightarrow r = \frac{-c + \sqrt{c^2-16}}{2} \Rightarrow c^2 = 16$

$\Rightarrow c = 4.$

(b) The amplitude (i.e. maximal value) of $y(x) = 3\cos(x) - 4\sin(x)$ is 5. [True / False]

$A = \sqrt{C_1^2 + C_2^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

(c) Every point $x_0 \in \mathbb{R}$ is an ordinary point of $(x^2-1)y''(x) + xy'(x) - (x+1)y(x) = 0$. [True / False]

$P(x) = x^2 - 1 = 0 \Rightarrow x = \pm 1.$

3. Second-Order Differential Equations

[3] (a) Consider the following differential equation:

$$y''(x) + 6y'(x) + 5y(x) = g(x).$$

Determine the complementary solution $y_c(x)$ then set-up, but *do not attempt to evaluate*, the trial form $y_p(x)$ for the given choices of $g(x)$ below.

$r^2+6r+5=0 \Rightarrow (r+5)(r+1)=0$

$\Rightarrow y_c(x) = C_1 e^{-x} + C_2 e^{-5x}$

(i) $g(x) = e^{-5x}\sin(x)$

$y_p = Ae^{-5x}\cos(x) + Be^{-5x}\sin(x)$

(ii) $g(x) = xe^{-x}$

$y_p(x) = Ax^2e^{-x} + Bxe^{-x}$

[4] (b) Given that $y_1(x) = 1 + x$ and $y_2(x) = e^x$ are fundamental solutions of $xy''(x) - (1+x)y'(x) + y(x) = 0$, determine the particular solution $y_p(x)$ of

$$xy''(x) - (1+x)y'(x) + y(x) = 2x^2e^x.$$

[*Hint:* Remember the standard form when applying variation of parameters!]

$W(y_1,y_2)(x) = y_1y_2' - y_1'y_2$

$= (1+x)e^x - e^x(1) = xe^x$

Note: $g(x) = \frac{2x^2e^x}{x} = 2xe^x$

$u_1 = -\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx = -\int \frac{e^x(2xe^x)}{xe^x}dx$

$= -2\int e^x dx = -2e^x$

$u_2 = \int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}dx = \int \frac{(1+x)(2xe^x)}{(xe^x)}dx$

$= 2\int(1+x)dx = 2x + x^2$

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$= -2e^x(1+x) + (2x+x^2)e^x = \underbrace{-2e^x}_{\text{in } y_c!} + x^2e^x = x^2e^x$

4. Power Series Solutions

[2] (a) Determine the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{n!} x^n$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(n+2)}{(n+1)!} \frac{n!}{(-1)^{n+1}(n+1)}\right|$$

$$= \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0$$

$\Rightarrow$ Converges everywhere

[2] (b) Write the following expression as a single summation whose common term is x^n:

$$x\sum_{n=1}^{\infty} n(n+1)a_n x^{n-2} + \sum_{n=2}^{\infty}(n-1)a_n x^{n-2}$$

$$= \sum_{n=1}^{\infty} n(n+1)a_n x^{n-1} + \sum_{n=0}^{\infty}(n+1)a_{n+2}x^n$$

$$= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+1} + (n+1)a_{n+2}\right]x^n$$

[2] (c) Suppose that a particular differential equation has a power series solution centered at $x_0 = 0$ generated by the recursion relation

$$a_{n+2} = \frac{a_{n+1} + a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Determine up to the x^3 term of the solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$n=0 \Rightarrow a_2 = \frac{a_1}{2} + \frac{a_0}{2}$$

$$n=1 \Rightarrow a_3 = \frac{a_2 + a_1}{6} = \frac{\left(\frac{a_1}{2} + \frac{a_0}{2}\right) + a_1}{6} = \frac{1}{12}a_0 + \frac{1}{4}a_1$$

$$\Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)x^2 + \left(\frac{1}{12}a_0 + \frac{1}{4}a_1\right)x^3 + \dots$$

5. Laplace Transforms:

[2] (a) Evaluate the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2+9}\right\}$$

$$= e^x \cos 3x + e^x \sin 3x$$

[4] (b) Use the Laplace transform method to solve the following initial value problem:

$$2y''(x) - 3y'(x) + y(x) = 0, \quad y(0) = 0, \ y'(0) = 1/2.$$

Laplace transform:

$$2[s^2Y(s) - s\underset{0}{y(0)} - \underset{\frac{1}{2}}{y'(0)}] - 3[sY(s) - \underset{0}{y(0)}] + Y(s) = 0$$

$$\Rightarrow [2s^2 - 3s + 1]Y(s) = 1$$

$$\Rightarrow Y(s) = \frac{1}{2s^2-3s+1} = \frac{1}{(2s-1)(s-1)} = \frac{A}{2s-1} + \frac{B}{s-1}$$

$$\Rightarrow 1 = A(s-1) + B(2s-1)$$

$$s = \tfrac{1}{2} \Rightarrow A = -2, \quad s = 1 \Rightarrow B = 1$$

$$\Rightarrow Y(s) = \frac{-2}{2s-1} + \frac{1}{s-1} = -\frac{1}{s-\frac{1}{2}} + \frac{1}{s-1}$$

$$\Rightarrow y(x) = -e^{\frac{1}{2}x} + e^x$$