MATH 319, WEEK 1:
Introduction, Definitions & Review

1 Introduction

At its heart, the study of differential equations is the study of *mathematical modeling*. More concretely, when investigating natural phenomena (e.g. motion of physical objects, interactions between competing species in an ecosystem, outbreaks of disease, etc.) it is often desireable to build a simplified mathematical model rather than performing repeated statistical studies and experiments.

It turns out that the modeling choice of differential equations is very natural for many applications, and so the techniques developed in this course will be very important in understanding the mathematical basis of many practical problems in the sciences. It should be pointed out, however, that differential equations are interesting mathematical objects of study in their own right, and can be studied with no reference to application at all. The theoretical study of differential equations will be touched upon in this course, but *will not* be a primary focus of study.

Throughout this course we will study the following topics (roughly Chapters 1, 2, 3, 5, 6, and 7 of the text):

- First-Order Differential Equations (Four Weeks)

- Second-Order Differential Equations (Three Weeks)

- Series Solutions to Differential Equations (Two Weeks)

- Laplace Transform Methods (Two Weeks)

- Systems of Differential Equations (Four Weeks)

We have heard now about how important differential equations are, so we had better stop for a moment and make sure we understand a *differential equation* is. And perhaps just as importantly, we should discuss some common avenues by which they arise. So, let's ask the following questions:

1. What is a differential equation?

2. How do differential equations arise in practice?

The simple answer to the first question is the following.

Definition 1.1. *A differential equation is any equation (i.e. algebraic expression) which involves functions and at least one of their derivatives.*

That's it! Basically, if you see an equation and it has a derivative (i.e. differential) in it, it is a differential equation.

Of course, not all differential equations are created equal. Many different classifications of differential equations exist and arise in different contexts throughout the sciences. Among the most important distinctions we will need to make between different equations are the following:

- **Order**: The *order* of a differential equation is the order of the highest-order derivative appearing in the expression. For example, $y'' = xy$ is second-order, $y' - \sin(x) = 0$ is first-order, and so on.

- **Linearity**: A differential equation is said to be *linear* if it is linear in all of its dependent variables and derivatives (i.e. dependent variables are *first power* and *isolated*!). **NOTE:** Linear DEs may be non-linear in their *independent* variables! For example, $y'' - 4y = \sin(x)$ and $y'' + (1/x)y' + x^2y = 0$ are linear while $(y')^2 = xy$ is not.

- **Ordinary** versus **partial** differential equations: A differential equation is said to be an *ordinary differential equation* (ODE) if all derivatives are taken with respect to the same variable. For example

$$\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$$

  is an ODE while

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

  is a *partial differential equation* (PDE).

The differences between the DEs above may seem small at first glance, but we will see that the methods employed to analyse these systems vary greatly as we cross these lines of classification. In this course, we will only consider *ordinary* differential equations and will start by considering equations which are *low order* and *linear*. As we expand our focus (e.g. increasing the order, consideration of nonlinear equations...) we will see that we will need to consider new and sometimes very sophisticated tools.

We now consider the second question: How do differential equations arise? In fact, differential equations can be formulated very simply. For example, we all know from basic calculus that the expression

$$y = \sin(x)$$

can be differentiated to give

$$\frac{dy}{dx} = \cos(x). \tag{1}$$

If we look at this for just a moment, however, we notice that (1) exactly fits into the definition we have just given. It is a differential equation! (Although a rather trivial one.) We can furthermore classify it is first-order, linear, and ordinary—the easiest case imaginable.

We can also get a sense from this example of what it means to *solve* a differential equation.

Definition 1.2. *A **solution** to a differential equation is any function which satisfies the given equation.*

That is to say, if we are given a differential equation, and a function (say $y(x)$) which we propose as a solution, we can check by plugging it into the differential equation. If the left-hand side and the right-hand side are equal, we are done.

For our previous example, we can clearly see that that $y(x) = \sin(x)$ is a *solution* of (1) because the left-hand side and right-hand side are equal. (In fact, it is not the only such choice, but we will get to that later.) This would have been obvious even if we had only been given (1) because we could directly *integrate* the expression to recover the function $y(x)$ (since the First Fundamental Theorem of Calculus guarantees that integration *undoes* differentiation). That is to say, integrating (1) gives

$$\int \frac{dy}{dx} \, dx = \int \cos(x) \, dx$$

$$\implies \quad y(x) = \sin(x) + C$$

so that, choosing $C = 0$, we were directly able to go from the differential equation form (1) to the desired solution $y(x)$.

Example: Show that $y = \tan(x - C)$ is a solution to the differential equation $y' = 1 + y^2$ for any value of $C \in \mathbb{R}$ (the symbol $\mathbb{R}$ means the real

numbers, that is to say, any number in the interval $(-\infty, \infty)$).

Solution: All we need to do is *check* that the right-hand and left-hand sides of the equation are equal. We have

$$\text{LHS} = \frac{dy}{dx} = \frac{d}{dx}\left[\tan(x - C)\right] = \sec^2(x - C)$$

and

$$\text{RHS} = 1 + y^2 = 1 + \tan^2(x - C) = \sec^2(x - C).$$

Since we have LHS=RHS, we are done!

Note: Whenever I give a differential equation and a solution and ask you to *check* whether the solution works, it is sufficient to *plug the function into the equation.* You do not need to solve the equation! (This can be a time-saver on exams, since it is easier to check answers than to derive them!)

It is important to stop here to make a very important note about solving differential equations. We will see that all of the integration techniques considered in previous courses (integration by substitution, integration by parts, trigonometric substitution, integration of rational functions, etc.) will be very important in understanding and solving differential equations. **These topics will be considered background knowledge and will not be reviewed in this course!** If you struggled with those topics in your previous calculus courses, it is very important to review them as soon as you can. They will be very important throughout this course!

That said, it turns out that integration is not *sufficient* for solving differential equations in general. In fact, differential equations arising in practice cannot be solve by simple integration. To see why this the case, let's consider a more physically motivated example.

Many real-world applied phenomena are understood by their *rates* of movement (or rate of rate of movement, i.e. acceleration, etc.). The most readily available example is Newton's second law of motion, which says that the force exerted on an object is equal to its mass times it acceleration, i.e.

$$F = ma. \tag{2}$$

We all know that an objections acceleration is the rate of change (i.e. derivative) of its velocity, which is the rate of change (i.e. derivative) of the object's

position, so that the acceleration is the second derivative of the object's position. In other words, we have

$$ma = m\frac{d^2x}{dt^2}.$$

That clarifies the right-hand side of (2), but what about the left-hand side? Depending on the application, different terms are used to represent the forces acting on a body. One simple assumption, which is used commonly in simple models of springs (via Hooke's law) or pendulums (as a result of gravity) is to assume that there is a **restoring force** proportional to the object's distance from its resting position. This is common represented as $F(x) = -kx$ where $k > 0$. (Notice that if $x > 0$, i.e. if the object is to the right of its resting position, then there is a restoring force pushing to the left; conversely, if $x < 0$, i.e. if the object is to the left of its resting position, then there is a restoring force pushing to the right.)

Combining this together into a equation via (2), we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \implies \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{3}$$

This is certainly a differential equation (it involves the function $x(t)$ and one of its derivatives, in this case the second derivative) but it **cannot be solved directly by integration**. To see why, recall that in order to integrate we need to have a function of the independent variable (in this case, t). In this case, however, we have the unknown function $x(t)$. We cannot integrate over t because we do not know what $x(t)$ is! In fact, determining what $x(t)$ is is exactly what we are trying to ascertain.

Nevertheless, we can still sensibly ask the question of what a solution to (3) might look like. All we are asking for is to find a function $x(t)$ which satisfies the expression. It should not take much convincing that there are several options. The easiest to check are $x_1(t) = \sin\left(\sqrt{\frac{k}{m}}t\right)$ and $x_2(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$. In fact, any solution of the form

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants will work. These solutions were not obtained using integration, however (although we will not get to the general method which was used for a few weeks yet).

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There are many other examples of simple differential equations which arise from the sciences which cannot be solved directly by integrating, including:

- **Exponential growth (populations)**

$$\frac{dP}{dt} = rP,$$

- **Logistic growth (populations)**

$$\frac{dP}{dt} = rP(K - P),$$

- **Newton's law of cooling**

$$\frac{dT}{dt} = k(T_{ext} - T),$$

- **Restoring with friction and forcing (second-order linear)**

$$\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f(t).$$

This gives us some sense of the kind of question we are going to be interested in during this course. We are going to be interested in the following questions:

1. Given a differential equation, is there a explicit solution (i.e. a function which satisfies the expression)? And if so, how can we find it?

2. How do we interpret solutions to differential equations in the context of the governing equations and/or the original physical motivation for them?

2   Slope/Direction Fields

So far, we have seen a number of examples of differential equations explicitly involving *equations* and only equations. We might wonder if there is a way to *visualize* the solutions and make sense out of them in the context of some sort of picture. That is to say, is there some *big picture* which binds these equations together?

To consider how we might approach this problem, let's consider the following general first-order ODE

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

Consider the following intuition:

1. Suppose $y(x)$ is a solution of (4).

2. Suppose (x, y) is a point on the curve $y(x)$.

3. Since $y(x)$ is a solution of (4), we have LHS=RHS so that the slope at (x, y) is given by $f(x, y)$. That is to say, the function $f(x, y)$ gives the slopes of the *local tangent lines* to the solution $y(x)$.

The power of this observation is that it holds *even if the exact form of $y(x)$ is not known*! We know what $f(x, y)$ is (it is assumed to be given). Computing values for various points (x, y) in the plane must give the slope for any possible solution passing through the point. If we do this for enough points, we should get a picture which tells us what the solutions look like! The resulting diagram is called a *direction* or *slope field*.

Example 1: Construct a slope field for the differential equation

$$\frac{dy}{dx} = \cos(x).$$

Solution: We already saw that this equation had solutions of the form $y(x) = \sin(x) + C$ for any value of $C \in \mathbb{R}$. We now want to construct a picture.

We notice that the RHS of the equation only depends on x, because we have $f(x, y) = \cos(x)$. In order to determine the slope values for the tangent lines, it is sufficient therefore to just consider a sample of values of x. We might as well pick the easiest values we can. We can see that

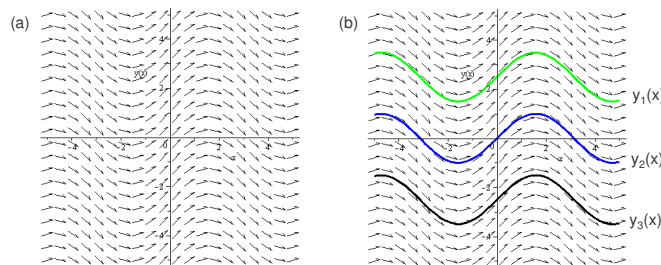| x	$\cos(x)$
$-\pi$	-1
$-\pi/2$	0
0	1
$\pi/2$	0
π	-1

Figure 1: In (a), the slope field of the differential equation $y' = \cos(x)$ is displayed. In (b), the solutions $y_1(x) = \sin(x) + 2.5$, $y_2(x) = \sin(x)$, and $y_3(x) = \sin(x) - 2.5$ overlain.

and that the sequence of values repeats from there. When we consider the (x, y)-plane we arrive at a picture like that contained in Figure 1. We can now see exactly how the solutions $y(x) = \sin(x) + C$ fit into the bigger picture! We have different solutions depending on which points we choose to have the solution pass through. The notion of having *families* of solutions to differential equations will be a recurring one. Also notice that we could have guessed the form (or at least the flavor) of the individual solutions just from the slope field.

Example 2: Construct a slope field for the differential equation

$$\frac{dy}{dx} = 1 + y^2.$$

Solution: Again, we already know that this has the solution $y(x) = \tan(x + C)$ for any $C \in \mathbb{R}$. Now we want to construct the slope field.

We notice that, once again, the RHS of the equation only depends on a single variable; however, in this case, we have that $f(x, y) = 1 + y^2$ only depends on y. At any rate, we can see that

y	$1 + y^2$
-2	5
-1	2
0	1
1	2
2	5

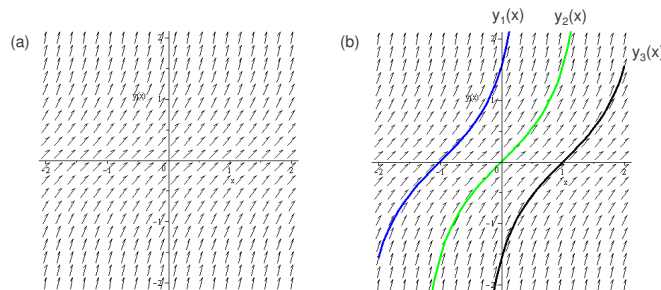In general, we have that the tangent lines become steep and steep the far-

Figure 2: In (a), the slope field of the differential equation $y' = 1 + y^2$ is displayed. In (b), the solutions $y_1(x) = \tan(x + 1)$, $y_2(x) = \tan(x)$, and $y_3(x) = \tan(x - 1)$ overlain.

ther away from $y = 0$ they are. This gives rise to the picture contained in Figure 2. Again, we can now clearly see how the analytic solutions fit into the bigger picture!

Example 3: Construct a slope field for the differential equation

$$\frac{dy}{dx} = x^2 + y^2.$$

Solution: This example is different than the previous ones in two important ways. Firstly, the RHS depends on both x and y, so it will not be sufficient to consider an assortment of values of just x or y. We will need to consider an assortment of points in the whole (x, y)-plane. Secondly, we do not know the explicit solution of the DE; in fact, no explicit solution involving elementary functions (x^2, $\sin(x)$, $\ln(x)$, e^x, etc.) exists!

We could start by picking a variety of points in the (x, y) plane and computing the value of $f(x, y) = x^2 + y^2$. This is what your computer does; however, we cannot do computations as quickly as our computers. Rather, we will seek a simpler method. Consider the following logic:

1. For fixed values $C^2 = x^2 + y^2$, the value of C corresponds to the radius of a circle centered at $(0, 0)$.

2. It follows from the DE that C^2 corresponds to the slope of the solutions through the corresponding points on the circle.

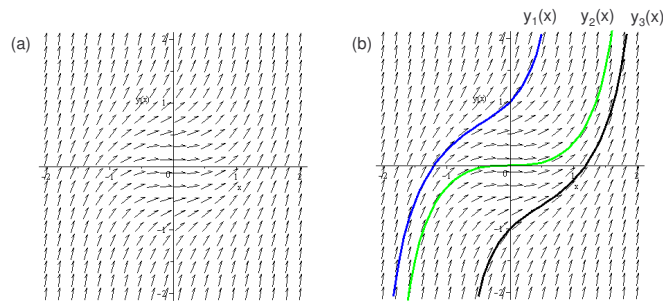That is to say, the slope of any solution through a point on the circle of

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Figure 3: In (a), the slope field of the differential equation $y' = x^2 + y^2$ is displayed. In (b), three solutions passing through the initial points $y_1 : (0, 1)$, $y_2 : (0, 0)$ and $y_3 : (0, -1)$ are overlain.

radius C is C^2. For instance, we have that

$$
\begin{aligned}
\text{radius} &= 0 &\implies& \quad \text{slope} = 0 \\
\text{radius} &= 1/2 &\implies& \quad \text{slope} = 1/4 \\
\text{radius} &= 1 &\implies& \quad \text{slope} = 1 \\
\text{radius} &= 3/2 &\implies& \quad \text{slope} = 9/4 \\
\text{radius} &= 2 &\implies& \quad \text{slope} = 4
\end{aligned}
$$

Putting everything together gives the picture contained in Figure 3. The really important thing to notice about this example is that, even though we do not have access to an explicit solution $y(x)$, we can very clearly see what solutions look like!