MATH 319, WEEK 2:
Initial Value Problems, Existence/Uniqueness, First-Order Linear DEs

1 Initial-Value Problems

We have seen that differential equations can, in general, given rise to *multiple* solutions. This should be reasonable disconcerting at first glance. After all, we imagine differential equations as representing some sort of physical phenomenon, and when we throw a projectile, or release a pendulum, or connect an electrical circuit, we do not observe *multiple* solutions. We observe exactly one. So how do we resolve the mathematical peculiarity of multiple solutions with the physical observation that only one thing can happen at a time?

The answer is that we define the differential equation *together* with the relevant *initial conditions*.

Definition 1.1. *The* ***initial-value problem*** *(IVP) associated with a first-order differential equation is the problem of solving*

$$\frac{dy}{dx} = f(x, y), \qquad \textit{subject to} \quad y(x_0) = y_0$$

where $x_0, y_0 \in \mathbb{R}$.

There are a few notes worth making here:

- The initial-value problem corresponds to picking the single trajectory which goes through the point (x_0, y_0) in the slope field diagram! We now know exactly how to fill out the slope field diagram with solutions.
- The terminology *initial-value* is chosen to reflect the reality that we are usually interested in centering the problem at zero (i.e. setting $x_0 = 0$). In problems where time is the independent variables, we have $t_0 = 0$, which is truly the initial value. We can, however, choose x_0 equal to another value (e.g. conditions like $y(3) = -7$ or $y(-1) = 10$).

- In general, we need as many initial conditions as we have constants in the general solution. For second-order differential equations, we will typically need *two* initial conditions, one on the variable itself, and one on the derivatives. For instance, for gravitational force problems where $x(t)$ is the height of an object, we need

$$x'' = -g, \quad \text{subject to} \quad x(t_0) = x_0,\ x'(t_0) = v_0$$

  to fully determine the solution to the initial value problem.

- A solution to a differential equation is called a **general solution** if it encapsulates all possible solutions to the corresponding initial-value problems. A solution is called a **particular solution** if it is associated to a specific initial value problem.

Example 1: Solve the initial value problem

$$\frac{dy}{dx} = y, \quad y(0) = 3.$$

Solution: We already know that the general solution of the differential equation is $y(x) = ke^x$ where $k \in \mathbb{R}$. It only remains to consider the initial condition $y(0) = 3$. Plugging in $x = 0$ gives us

$$y(0) = 3 = ke^{(0)} \quad \Longrightarrow \quad k = 3.$$

It follows the the particular solution we are interested in is $y(x) = 3e^x$.

Example 2: Consider a projectile thrown up into the air from the top of a cliff which is 50 meters from the ground. Suppose the projectile is subject only to the force of gravity ($F = -mg = -9.8m$ kg·m/s^2) and suppose the initial upward velocity of the throw is 10 m/s. Solve the initial value problem. How long does it take the projectile to reach the bottom of the cliff?

Solution: From Newton's second law, we have that $F = ma$ so that

$$-mg = mx''.$$

With the given information, and removing the dimensions (which fortunately do match up) we can restate this as an initial value problem as

$$x'' = -9.8, \quad x(0) = 50,\ x'(0) = 10.$$

This can be directly integrated to get

$$x'(t) = \int \frac{d^2x}{dt^2}\, dt = -\int 9.8\, dt = -9.8t + C.$$

We can now use the first piece of initial information to get

$$x'(0) = 10 \quad \Longrightarrow \quad 10 = -(9.8)(0) + C \quad \Longrightarrow \quad C = 10.$$

It follows that we have

$$x'(t) = -9.8t + 10.$$

We can integrate this again to get

$$x(t) = \int \frac{dx}{dt}\, dt = \int (-9.8t + 10)\, dt = -4.9t^2 + 10t + D.$$

The other piece of initial information gives us

$$x(0) = 50 \quad \Longrightarrow \quad 50 = -4.9(0)^2 + 10(0) + D \quad \Longrightarrow \quad D = 50.$$

It follows that the solution to the initial value problem is

$$x(t) = -4.9t^2 + 10t + 50.$$

As we might have expected, this is a parabola opening down. The vertex corresponds to the maximum height before it starts its descent to the ground. To answer the final question, we recognize that reaching the ground corresponds to setting $x = 0$. It follows that we need to find a time such that

$$-4.9t^2 + 10t + 50 = 0.$$

The quadratic formula gives the solutions $t = -2.33$ and $t = 4.37$. We can reject the negative value since it occurs before we release the projectile and conclude that the projectile will reach the ground in 4.37 seconds.

2 Existence and Uniqueness of Solutions (Section 2.8 in Text)

So far we have developed an intuition on what it means to be a solution of a differential equation, how to check if a function is in fact a solution, and how to interpret solutions geometrically. We have not, however, given any consideration to the following far more basic questions:

1. Does a solution always exist? And if it exists, does it exist everywhere?
2. If we have a solution, is it necessarily unique?

At first glance, there questions may seem a little absurd. We are asking whether we should even consider the question of finding a solution in the first place!

It turns out there is a significant amount of subtlety involved. Consider the following examples.

Example 3: If possible, determine a solution of

$$\left(y'\right)^2 + y^2 = -1.$$

Solution: At first glance, this seems like a sensible problem. It is clearly a differential equation—first-order and ordinary, like many we have seen already. In principle, if somebody proposed a function $y(x)$, we could check in the equation to verify whether it was indeed a solution or not. It is only when we dig a little deeper that we see something is terribly, horrendously wrong. We might notice firstly that, if we were asked to solve the *algebraic* equation

$$x^2 + y^2 = -1$$

would would immediately reject the question as senseless. The LHS is necessarily positive while the RHS is clearly negative—no solution exists. There is no difference when we consider the *differential* equation given! We do not need to attempt to find a solution in order to know one does not exist. **Differential equations are not guaranteed to have even a single solution**. (Although most of the differential equations we will consider in this course will have solutions!)

Example 4: We know that $y(x) = \tan(x)$ is a solution of $y' = 1 + y^2$ (see Figure 1). We might, however, notice something strange about it: it is not connected! We encounter a rather abrupt jump when we hit $\pi/2$ and then switch instantaneously from $+\infty$ to $-\infty$. This is not a significant concern to our mathematical analysis (everything we have done is correct!) but it might be a concern to the physical problem were are modeling. For instance, suppose we are modeling the position of some object—we cannot very well have the object explode to infinity and wrap around the other side. In applied examples we will be careful to consider only connected (i.e. continuous) portions of solutions, lest we run into such absurdities. At any rate, this is a point worth emphasizing. **Solutions, even if they exist,**

are not guaranteed to exist everywhere!

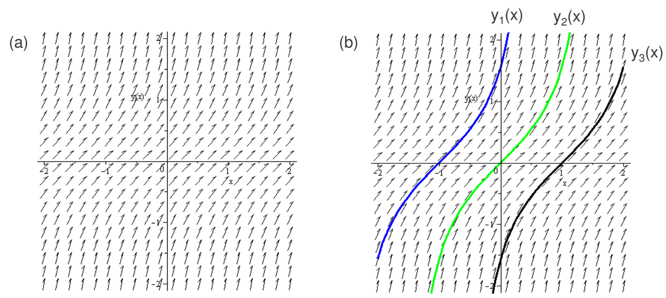

Figure 1: Slope field of $y' = 1 + y^2$ with $y_1(x) = \tan(x+2.5)$, $y_2(x) = \tan(x)$, and $y_3(x) = \tan(x - 2.5)$ overlain. Solutions are continuous only a finite interval.

Example 5: Show that, for any $C \in \mathbb{R}$,

$$y(x) = \begin{cases} 0, & x \leq C \\ (x-C)^2, & x > C \end{cases}$$

is a solution of $y' = 2\sqrt{y}$. Comment on the uniqueness of solutions.

Solution: We have that $y = 0$ implies $y' = 0$ and $\sqrt{y} = 0$ trivially so $y = 0$ always satisfies $y' = 2\sqrt{y}$. To the other half of the proposed solution, we have

$$\frac{dy}{dx} = 2(x - C)$$

and

$$2\sqrt{y} = 2\sqrt{(x-C)^2} = 2|x + C| = 2(x - C)$$

where we have removed the absolute value because $x > C$ implies $x - C$, which implies $|x - C| = x - C$. It follows that both halves of the expression satisfy the differential equation, and since we have continuity and equality in derivatives at $x = C$, the function is smoothly defined at the transition. It follows that it is a solution.

We notice something a little strange when we try to consider the slope field, however (see Figure 2). The solution is a constant ($y = 0$) to the left of C and the right-half of a parabola to the right of C, but when does the transition happen? Suppose we are the point $(0, 0)$ and are travelling along the solution $y = 0$ to the right. How do we choose when we branch off to the parabola? Or even if we do? We have that $y = 0$ is always a solution, after all, so why even bother considering the parabolic answer?

The problem is that solutions *overlap* at $y = 0$. That is to say, they are not separated, as they were in the previous examples. Every solution with $C \geq 0$, for instance, goes through the point $(0, 0)$. So not only can we have solutions be non-unique due to the existence of a family of solutions, we can have them be non-unique when we restrict to solutions through a single point in the solution space as well (although this is uncommon!).

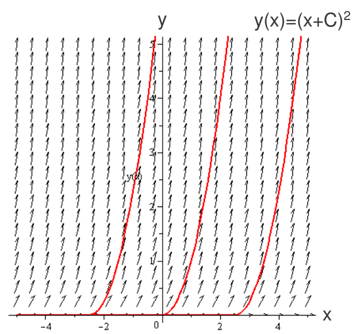

Figure 2: Slope field of $\dfrac{dy}{dx} = 2\sqrt{y}$ with $y = 0$ and the right-halves of $y_1(x) = (x + 2.5)^2$, $y_2(x) = x^2$, and $y_3 = (x - 2.5)^2$ overlain. Every solution with $C > x$ goes through the point $(x, 0)$ so that solutions intersect.

This raises an interesting question: Without prior knowledge about a solution, can be guaranteed a solution exists and/or that it is unique? The answer (fortunately) is *yes* and is the content of the following theorem.

Theorem 2.1 (Theorem 2.8.1 in Text)**.** *Consider the first-order ODE $y' = f(x, y)$ and an initial point (x_0, y_0). Let $\mathcal{R}$ denote a non-empty region around the point (x_0, y_0). Then:*

1. *if $f(x, y)$ is continuous in $\mathcal{R}$, then there is a subregion $\mathcal{R}' \subseteq \mathcal{R}$ also containing (x_0, y_0) in which there is a solution of the DE through (x_0, y_0);*

2. *if, furthermore, $\partial f/\partial y$ is continuous in $\mathcal{R}$, then there is a subregion $\mathcal{R}' \subseteq \mathcal{R}$ also containing (x_0, y_0) in which there is a* unique *solution of the DE through (x_0, y_0).*

There are a few things worth noting about this theorem:

- The subregion of existence (and/or uniqueness) is the *bare minimum* guaranteed by the Theorem. That is to say, it is quite possible that solutions exist for a much broader region of the (x, y)-plane, even though the Theorem does not guarantee it. In fact, it is normally the case that solutions only cease to exist in very, very small regions.
- An important element of Theorem 2.1 is that we have the DE in the form $y' = f(x, y)$. In other words, we must solve for y' and have it isolated in the expression. The Theorem is also insufficient to consider cases of *higher-order* DEs or PDEs. These equations have their own theorems!
- This theorem is most powerful when we *do not know the actual solution* $y(x)$! As we will become increasingly interesting in finding solutions as the course progresses, we will not consider questions of existence and uniqueness very often. In the general study of differential equations, however, the question of existence and uniqueness is a very challenging one which is still the source of significant research.

Examples: Use Theorem 2.1 to make claims about the existence and uniquess of the three examples considered above, namely, $(y')^2 + y^2 = -1$, $y' = 1 + y^2$, and $y' = 2\sqrt{y}$.

Solution: The first example must be rewritten into the form $y' = f(x, y)$. We have that

$$(y')^2 + y^2 = -1 \quad \Longrightarrow \quad y' = \pm\sqrt{-1 - y^2}$$

so that $f(x, y) = \sqrt{-1 - y^2}$. Since this clearly is not continuous for any (x, y)—it does not even exist!—we may conclude no solutions exist at all.

The second example is already in the form $y' = f(x, y)$. We can furthermore see that

$$f(x, y) = 1 + y^2$$

and

$$\frac{\partial f}{\partial y} = 2y$$

are both continuous. The Theorem therefore allows us to conclude that there is a unique solution through every point in the (x, y)-plane. (Note that, although we found earlier that the solutions exist on finite intervals only, it is true that every point has a unique solution through it!)

The third example is also already in the form $y' = f(x, y)$. We have that

$$f(x, y) = 2\sqrt{y}$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}.$$

We notice that part 1. of the theorem guarantees the existence of solutions through all points $y \geq 0$ but part 2. only guarantees *uniqueness* of solutions through points satisfying $y > 0$. This is exactly what we found with the explicit solution! Solutions are well separated in the positive half-plane but bunch up when $y = 0$. Even through the solutions exist there, we are not able to distinguish between them.

3 First-Order Linear Equations

Consider the first-order differential equation

$$\frac{dy}{dx} = \frac{1 - y}{x}.$$

We are now interested in solving this DE from first principles. In other words, we want to *find* a function $y(x)$ which satisfies the expression when none is given to us. How can we do this?

We might notice that we can rewrite the expression as:

$$x\frac{dy}{dx} + y = 1.$$

There is nothing in the expression dictating that we have to do this (yet!) but we can notice at least one nice thing about this form: it was easy to classify! Everything involving y and its derivatives is isolated (with respect to terms involving y), so it is a **first-order linear differential equation**.

There is a little bit of "cheating" that has been done in rearranging the expression this way, but it is a suggestive bit of cheating. Let's consider just the left-hand side of the above expression, i.e.

$$x\frac{dy}{dx} + y.$$

If we stare this for long enough, or were born with unparalleled mathematical powers, we might notice that this can be written in a more compact form. Without justifying, for a moment, why we would *want* to do this, we might notice that this expression is the end result of the product rule for differentiation on the term xy. That is to say, we have

$$\frac{d}{dx}[xy] = x\frac{dy}{dx} + y.$$

In other words, we can take the two terms on the left-hand side and condense them into a single term, at the expense of having to recall the product rule for differentiation. We can now rewrite the differential equation above as

$$\frac{d}{dx}[xy] = 1.$$

It should take far less mathematical insight to recognize that this is a *huge* improvement over our previous expression. The reason should be clear: we can integrate it! If we integrate the left-hand and right-hand sides by x, the Fundamental Theorem of Calculus tells us the differential on the left-hand side disappears, and the right-hand side can be evaluated as long as we know an anti-derivative of whatever the term there happens to be. That is to say, we have

$$\int \frac{d}{dx}[xy]\ dx = \int 1\ dx$$
$$\implies \quad xy = x + C, \quad C \in \mathbb{R}$$

which, after dividing by x, implies that we have the general solution

$$y(x) = 1 + \frac{C}{x}, \quad C \in \mathbb{R}.$$

It can be easily verified that this is in fact a solution of the DE (check!).

At this point, we should feel a little excited. We are on the path toward discovering a method for solving first-order linear differential equations. So far, the steps we took were:

1. Write with y and y' on one side,
2. Combine term on left by reversing the product rule,
3. Integrate,
4. Solve for y.

We will see in a few minutes that this is not sufficient to solve all first-order linear differential equations, but the intuition—especially the trick with the product rule—will prove to be the key to the general method.

Now consider the example

$$x\frac{dy}{dx} + 2y = 1.$$

This is only subtly different that the previous example—in fact, the only difference is the coefficient of the y term is now two. This subtle difference, however, is enough to sabotage our earlier intuition with regards to a solution method, since there is no function $f(x)$ such that

$$\frac{d}{dx}\left[f(x)\ y\right] = x\frac{dy}{dx} + 2y.$$

So what can we do?

Let's consider changing the expression (again!) but in a different way. Let's consider *multiplying* through by a single term that is a function of x. In this case, let's choose the function to be x itself. This gives us

$$x^2\frac{dy}{dx} + 2xy = x.$$

If there were any questions with regards to *why* we would want to do that, I hope they have now been answered. Using our earlier intuition with regards to the product rule, we can clearly see that we have

$$\frac{d}{dx}\left[x^2 y\right] = x^2\frac{dy}{dx} + 2xy = x.$$

Again, we can integrate to get the solution. We have

$$\int \frac{d}{dx}\left[x^2 y\right]\ dx = \int x\ dx$$

$$\implies \quad x^2 y = \frac{x^2}{2} + C, \quad C \in \mathbb{R}$$

so that the desired solution is

$$y(x) = \frac{1}{2} + \frac{C}{x^2}, \quad C \in \mathbb{R}.$$

So what was different about this example? The difference was that we had to *multiply* by some factor before we could use the product rule trick that we just discovered to get to a form we could integrate. This multiplicative factor is called an **integration factor** and is generally denoted $\mu(x)$. We still have to wonder how we could find integration factors. After all, how did I know to multiply by the factor x?

It is perhaps best now to scale back and consider **first-order linear systems** at their most general. In general, we have

$$\frac{dy}{dx} + p(x)y = q(x). \tag{1}$$

This is only slightly different than the forms we have been using. We now want to get all the terms involving y on the left-hand side, and also to divide through by whatever the coefficient of the derivative is so that the derivative appears by itself. Now we ask the question: *What do we have to multiply by in order to guarantee that the two terms on the left-hand side can be combined using the product rule (in reverse)*?

The answer is not obvious at first glance, but it is easy to verify. The **integration factor** we need is

$$\mu(x) = e^{\int p(x)\,dx}.$$

The details are easy to check. We know by the Fundamental Theorem of Calculus and the chain rule that $\mu'(x) = p(x)\mu(x)$ so, if we multiply the entire expression by $\mu(x)$, we have

$$\mu(x)\frac{dy}{dx} + p(x)\mu(x)y = \mu(x)q(x).$$

The left-hand side can be simplified by noting that

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu(x)\frac{dy}{dx} + p(x)\mu(x)y.$$

It follows that the differential equation can be rewritten as

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)q(x).$$

We can then integrate to get

$$\mu(x)y = \int \mu(x)q(x)\,dx$$

and isolate y to get the general solution

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx \\ &= e^{-\int p(x)\,dx}\int \left(e^{\int p(x)\,dx}q(x)\right)\,dx. \end{aligned}$$

That's it! So long as we can evaluate these integrals, we can solve any first-order linear differential equation.

There are a few notes worth making:

- It is not necessary to include the arbitrary constant in the integration factor integral (i.e. take $C = 0$) or the absolute value for logarithms. Both cases amount to multiplying the expression by an arbitrary constant, which does not change anything.

- On the other hand, it is very important to remember to add the constant $+C$ to the other integration (resolving the product rule).

- It is important to have the equation in the form (1). Otherwise, the given integration factor will not work. In particular, notice that the coefficient of y' must be one.

- It is sufficient, but not recommended, to remember the general form of the solution. All of the steps in this derivation are based on tricks we know how to do, even if recognizing how to apply them might have been a little tricky.

- Whether these equations are autonomous or homogeneous depends on the forms of $p(x)$ and $q(x)$, although this particular method will work regardless.

Examples: Determine the integration factor $\mu(x)$ for the following differential equations and use it to find the general solution $y(x)$ and the particular solution for the given initial condition.

1. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x}, \quad y(1) = 1.$
2. $\dfrac{dy}{dx} + y = e^{-3x}, \quad y(0) = 2$
3. $(x+1)\dfrac{dy}{dx} - xy = e^{x}, \quad y(1) = 0.$

Solution (1): This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned} \mu(x) &= e^{\int p(x)\, dx} \\ &= e^{\int \frac{1}{x}\, dx} \\ &= e^{\ln(x)} = x. \end{aligned}$$

We can ignored the normally required $|x|$ in the $\ln(x)$ term by noticing that the two absolute value cases ($x > 0$ and $x < 0$) amount to multiplying the whole differential equation by a negative, which does not change it. Multiplying the entire expression by $\mu(x) = x$ gives us

$$x\frac{dy}{dx} + y = 1$$

which we have already seen. This was our original toy example. We already know that the general solution is

$$y(x) = 1 + \frac{C}{x}.$$

Substituting the intial value $y(1) = 1$ gives us

$$y(1) = 1 = 1 + C \quad \implies \quad C = 0.$$

It follows that the particular solution is

$$y(x) = 1.$$

Solution (2): This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned} \mu(x) &= e^{\int p(x)\ dx} \\ &= e^{\int 1\ dx} \\ &= e^x. \end{aligned}$$

Multiplying the entire expression by $\mu(x) = e^x$ gives us

$$e^x\frac{dy}{dx} + e^x y = e^x \cdot e^{-3x} = e^{-2x}.$$

Recognizing that the left-hand side now must be the product rule form (expanded out), we have

$$\frac{d}{dx}\left[e^x y\right] = e^{-2x}.$$

We could jump right to this if we wanted to, but it is important to recognize the intermediate step to check that we have determined the correct integration factor. We can integrate this to get

$$\int \frac{d}{dx}\left[e^x y\right]\ dx = \int e^{-2x}\ dx$$

$$\implies \quad e^x y = -\frac{e^{-2x}}{2} + C$$

$$\implies \quad y(x) = -\frac{e^{-3x}}{2} + Ce^{-x}.$$

Using the initial condition $y(0) = 2$ gives

$$y(0) = 2 = -\frac{1}{2} + C \quad \implies \quad C = \frac{5}{2}.$$

The particular solution is therefore

$$y(x) = -\frac{e^{-3x}}{2} + \frac{5e^{-x}}{2}.$$

Solution (3): This is not in standard form, so we need to do a little work. Dividing by $(x+1)$ we arrive at

$$\frac{dy}{dx} - \frac{x}{x+1}y = \frac{e^x}{x+1}.$$

In order to determine the integrating factor, we will need to determine the integral of $-x/(x+1)$. Using the substitution $u = x+1$, we have

$$-\int \frac{x}{x+1}\,dx = \int \frac{1-u}{u}\,du = \int \left(\frac{1}{u} - 1\right)\,du = \ln(u) - u = \ln(x+1) - (x+1).$$

Recognizing that constants (i.e. the -1) do not matter for integrating factors, we arrive at

$$\mu(x) = e^{\ln(x+1)-x} = (x+1)e^{-x}.$$

Multiplying the entire expression by $\mu(x) = (x+1)e^{-x}$ gives us

$$(x+1)e^{-x}\frac{dy}{dx} - xe^{-x}y = 1.$$

It follows that we have

$$\frac{d}{dx}\left[(x+1)e^{-x}y\right] = 1$$

which can be checked. Integrating with respect to x gives

$$(x+1)e^{-x}y = x + C$$

so that the general solution is

$$y(x) = \frac{e^x}{x+1}\left(x + C\right).$$

The initial condition $y(1) = 0$ gives

$$y(1) = 0 = \frac{e}{2}(1 + C) \quad \Longrightarrow \quad C = -1.$$

It follows that the particular solution is

$$y(x) = e^x \left(\frac{x-1}{x+1} \right).$$

Other examples are available in Section 2.1 of the text.