MATH 319, WEEK 3:
Separable DEs, Substitution Methods

1 Separable Equations (2.2 in text)

Re-consider the first-order ODE

$$\frac{dy}{dx} = \frac{1-y}{x}.$$

We previously showed that this equation could be rearranged into the standard form $y' + p(x)y = q(x)$ of a first-order linear DE. We could then find the solution by determining an integrating factor $\mu(x)$ and integrating.

Now we will consider another common method which can often be applied to first-order ODEs to change the problem into one of integration. This method depends on breaking the problems into *two* separate integration problems, in fact: one with respect to y, and one with respect to x. We notice that, as written, the right-hand side of the ODE depends on *both* x and y so we cannot integrate this directly with respect to x to determine the general solution. But we might notice that we can still make the problem "look like" an integration problem with respect to x by removing the y from the right-hand side and moving the differential dx to the other side. This leaves us with

$$\frac{dy}{1-y} = \frac{dx}{x}.$$

Furthermore, not only does the right-hand side look like an integral question (with respect to x), but the left-hand side looks like an integral question as well (with respect to y). In fact, that is exactly how we will treat the equation! When we integrate (with respect to y on the left, and x on the right), we obtain

$$\int \frac{1}{1-y}\, dx = \int \frac{1}{x}\, dx \implies -\ln|1-y| = \ln|x| + C \implies |1-y| = \frac{k}{|x|}$$

where $k = e^{-C} > 0$. There are a few technical details to sort out yet with the absolute value. In general, these will not be too important, but for completed we will fill in the details for this particular example. We have the following four cases:

$$y > 1, x > 0 \implies -(1-y) = \frac{k}{x} \implies y = 1 + \frac{k}{x}, k > 0$$

$$y > 1, x < 0 \implies -(1-y) = -\frac{k}{x} \implies y = 1 - \frac{k}{x}, k > 0$$

$$y < 1, x > 0 \implies (1-y) = \frac{k}{x} \implies y = 1 - \frac{k}{x}, k > 0$$

$$y < 1, x < 0 \implies (1-y) = -\frac{k}{x} \implies y = 1 + \frac{k}{x}, k > 0$$

Recognizing that $y = 1$ (i.e. $k = 0$) is a trivial solution, we have that the sign of k does not actually matter. The general solution is $y = 1 + \frac{k}{x}$ for $k \in \mathbb{R}$. This is exactly the answer we obtained when we solved this equation earlier!

There are a few notes worth making:

- The general trick we have performed is to separate all of the dependence on y on one side of the expression and all of the dependence on x on the other. Such differential equations are called **separable** and have the general form

$$f(y)\frac{dy}{dx} = g(x) \quad \text{or} \quad f(y)\; dy = g(x)\; dx.$$

- While everything "looks" good, we have been *very* lax in our justification of this separation (i.e. in "splitting" the differential, and integrating with respect to separate variables on the separate sides). A rigorous justification of the procedure can be made by application of the chain rule.

- It is a fairly general propery that a rigorous consideration of absolute value overcomes the seeming loss of negativity when we raise our arbitrary constants into the exponent. We will not perform the cases for further examples.

- I warned you that integration would be important for solving differential equations, and there is no class of systems that better exemplifies that than separable equations. Not only do we have to integrate to solve a separable equation, but in general we have to integrate *twice*.

Further examples are contained Section 2.2 of the textbook.

Example 2: Solve the initial value problem

$$\frac{dy}{dx} = -y^2\frac{(1+2x^2)}{x}, \quad y(1) = 1.$$

Solution: We can check very quickly that this cannot be manipulate into the form of a general first-order linearly differential equation. The reason is clear—the term y^2 is non-linear in y. It therefore does not fit into this class of equations.

Rather, we notice that, if we divide the equation by y^2, and move the differential dx to the right-hand-side, we have

$$\frac{1}{y^2}\, dy = -\frac{(1+2x^2)}{x}\, dx.$$

This is perfect! We have isolated the dependence on y on the LHS, and the dependence on x on the RHS. Now it is only a matter of integrating (twice!). We have

$$\begin{aligned}\int \frac{1}{y^2}\, dy &= -\int (\frac{1}{x} + 2x)\, dx \\ \implies -\frac{1}{y} &= -\ln(x) - x^2 + C \\ \implies y(x) &= \frac{1}{\ln(x) + x^2 + \tilde{C}}\end{aligned}$$

where $\tilde{C} = -C$. The initial condition $y(1) = 1$ gives

$$1 = \frac{1}{1+C}$$

so that $C = 0$. It follows that the particular solution is

$$y(x) = \frac{1}{\ln(x) + x^2}.$$

2 Substitution Methods

Many first-order differential equations do not fall directly within the classes of separable or first-order linear systems. Nevertheless, many common identifiable classes of differential equations can be manipulated into one of these two forms via the use of a carefully selected *variable substitution*. We will look at the following examples:

1. **(Power) Homogeneous equations:** Differential equations of the general form
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

A substitution of the form $v = \dfrac{y}{x}$ produces a *separable* differential equation in v and x of the form

$$x\frac{dv}{dx} = F(v) - v.$$

2. **Bernoulli equations:** Differential equations of the general form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

A substitution of the form $v = y^{1-n}$ produces a *first-order linear* differential equation in v and x of the form

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

There are a few notes worth making before we delve too deeply into these methods.

- While the given formulas are known and sufficient to solve most problems, as with first-order linear equations it is hoped that it is the *method* which is memorized, not the end formula. In other words, remember the required variable substitutions for the two types of equations.

- As with any problem involving variable substitutions involving derivatives, it is helpful to write out the tree of variable dependences. In particular, for the differential equations we are looking at, where we are looking for a function $y = y(x)$ (i.e. y as a function of x), if we define a variable transformation $v = v(x, y)$, we have the tree given in Figure 1, so that (according to the chain rule) the total derivative of v with respect to x is given by

$$\frac{dv}{dx} = \frac{\partial v}{\partial y}\frac{dy}{dx} + \frac{\partial v}{\partial x}.$$

3 (Power) Homogeneous Differential Equations

Consider the differential equation

$$2xy\frac{dy}{dx} = x^2 + y^2.$$

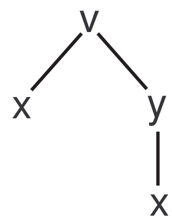

Figure 1: Variable dependence tree for $v = v(x, y)$, where $y = y(x)$.

It should not take much arguing to convince yourself that this differential equation is neither separable nor first-order linear. We need an alternative method for such differential equations.

One possibility is to choose an appropriate variable substitution. In this case, the necessary substitution is

$$v(x, y) = \frac{y}{x}.$$

With this variable substitution, we have

$$y = xv \quad \Longrightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

so that the differential equation can be rewritten in terms of the variables v and x as

$$\begin{aligned} 2x(xv)\left(x\frac{dv}{dx} + v\right) &= x^2 + (xv)^2 \\ \Longrightarrow \quad 2x^3v\frac{dv}{dx} &= x^2 + x^2v^2 - 2x^2v^2 \\ \Longrightarrow \quad 2x^3v\frac{dv}{dx} &= x^2(1 - v^2) \\ \Longrightarrow \quad \frac{2v}{1 - v^2}\, dv &= \frac{1}{x}\, dx. \end{aligned}$$

Why this substitution helps us should now be clear. While the differential equation was not easy to solve in the variables y and x, in the variables v and x it reduces to a separable differential equation, which is among the most straight forward classification of differential equations to identify and solve. We still have some work to do, however. Continuing, we have

$$\int \frac{2v}{1 - v^2}\, dv = \int \frac{1}{x}\, dx$$

$$\implies -\ln(1-v^2) = \ln(x) + C,$$
$$\implies 1 - v^2 = \frac{k}{x}.$$

Now that the integration step has been resolved, we would like to return to the original variables x and y. We started with $v = y/x$, so we now have

$$1 - \left(\frac{y}{x}\right)^2 = \frac{k}{x}$$
$$\implies \quad x^2 - y^2 = k\frac{x^2}{x} = kx.$$

We have the final general solution

$$y = \pm\sqrt{x^2 + \tilde{k}x}$$

where $\tilde{k} = -k$.

We should stop to make a few notes on this process.

- This differential equation belongs to a class of first-order differential equations called **(power) homogeneous differential equations**. Every (power) homogeneous differential equation can be written in the form
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$
by dividing by an appropriate power of x on the top and bottom of the previous form. The substitution $v = \frac{y}{x}$ is guaranteed to reduce such differential equations into a separable differential equation in v and x! (In other words, the technique we used in the example will *always* work, although, as we saw, we may still run into some tricky integration.)

- It is important to recognize differential equations which look (power) homogeneous but which in fact are not. For example, the differential equations
$$\frac{dy}{dx} = x + y$$
and
$$\frac{dy}{dx} = x^2 + 2xy + y^2$$
are *not* (power) homogeneous because there is denominator on the right-hand side with powers of x and y (the power is effectively zero, whereas the power of the numerator is two).

- I will make the distinction between homogeneous and *power* homogeneous differential equations. The reason for this is unfortunate: within the study of differential equations there are *two* accepted definitions of what constitutes a homogeneous differential equation, and these definitions are very different. Usually context will dictate which meaning is implied, but just to be clear I will attempt to use *power* homogeneous to refer to the class of differential equations we were just introduced to.

4 Bernoulli Differential Equations

We have seen how (power) homogeneous first-order differential equations can be transformed into separable equations by application of a fairly simple variable substitution. It turns out that there is a general class of differential equations which can be transformed into our other canonical solution class, first-order linear equations.

Consider the differential equation

$$3xy^2\frac{dy}{dx} = 3x^4 + y^3.$$

It should not take much arguing (again) to convince ourself that this equation is not separable, is not first-order linear, and is not even homogeneous (although it is close). Based on the methods we have established so far, we are basically stuck, but we are able not going to stop there. Let's try to rearrange this equation to get it as close to the first-order linear form as possible. We have

$$\frac{dy}{dx} = x^3y^{-2} + \frac{1}{3x}y \quad \Longrightarrow \quad \frac{dy}{dx} - \frac{1}{3x}y = x^3y^{-2}.$$

We actually have not done too poorly! In fact, it is only the term on the right-hand side that presents a problem. In particular, we are not happy with the y^{-2} and would like to make it go away.

Consider the substitution $v = y^3$. We want to rewrite this differential equation in y and x as a differential equation in v and x. This will require solving for the differential and all of the y terms. We have

$$y = v^{1/3} \quad \Longrightarrow \quad \frac{dy}{dx} = \left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx}$$

and $y^{-2} = v^{-2/3}$. It follows that the differential equation can be rewritten as

$$\left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx} - \frac{1}{3x}v^{1/3} = x^3v^{-2/3}.$$

Multiplying across by $3v^{2/3}$ we arrive at

$$\frac{dv}{dx} - \frac{1}{x}v = 3x^3.$$

When we look at this, we notice that, quite remarkably, the non-linear term has disappeared. This is a linear equation in v and x! We know how to solve these types of equations. We have the integration factor

$$\mu(x) = e^{-\int \frac{1}{x}\,dx} = e^{-\ln(x)} = \frac{1}{x}.$$

This gives us

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = 3x^2 \quad \Longrightarrow \quad \frac{d}{dx}\left[\frac{1}{x}v\right] = 3x^2$$

$$\Longrightarrow \quad \frac{1}{x}v = x^3 + C \quad \Longrightarrow \quad v = x^4 + Cx.$$

We are not, of course, completely done. The original questions was a differential equation with respect to y and x, so we need to change by to our original variables. We have

$$y^3 = x^4 + Cx \quad \Longrightarrow \quad y(x) = \sqrt[3]{x^4 + Cx}.$$

This was a rather remarkable solution method, but what intuition was underlying it? It turns out that this differential equation belongs to a class of differential equations called **Bernoulli differential equations**. We pause to make the following notes about them:

- The general form of a Bernoulli differential equation is

  $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

  and the required substitution is $v = y^{1-n}$. This is guaranteed to produce a first-order linear differential equation in v and x of the form

  $$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

- Notice that there are two values which are troublesome for this transform, $n = 0$ and $n = 1$. For $n = 0$ we have $v = y$, which is trivial, and for $n = 1$ we have $v = 1$, which is meaningless. Our concern, however, turns out to be very premature. Returning to the original form of the equations, we notice that $n = 0$ and $n = 1$ both correspond to a linear first-order differential equation in the first place (for $n = 0$ this is direct, and for $n = 1$ we just have to move the term on the right to the left-hand side).
- It is worth noting that this holds for *all* values of n other than $n = 0$ and $n = 1$. That is to say, we can consider fractional powers (e.g. $n = 1/2$, $n = 7/5$, $n = 92/13$, etc.) and negative powers ($n = -3$, $n = -4/9$, $n = -103$, etc.).

5 Exact Differential Equations

We saw that the trick for first-order differential equations was to recognize the general property that the product rule from differentiation yields, as if by design, a form that looks like a first-order linear equation. That is to say, we have

$$\frac{d}{dx}\left[f(x)y\right] = f(x)\frac{dy}{dx} + f'(x)y.$$

This certainly *looks like* a first-order linear differential equation—all we have to do is set this equation equal to something (potentially a function of x) and we are good to go. When we investigated these problems from the other direction, trying to reverse the product rule, we recognized that we were always able to do so after (potentially) multiplying by an appropriate *integration factor*.

We might realize that there is another differentiation operator which produces a very similar form. If we consider a general function $F(x, y)$, recognizing the dependence of y of x, we have from the chain rule that

$$\frac{d}{dx}\left[F(x,y)\right] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

This certainly looks like a first-order differential equation. The difference is that F_x and F_y are allowed to be functions of *both* x and y. Worse still, they are allowed to be *nonlinear* functions of y. At any rate, this forms a general class of differential equations known as **exact** differential equations. They have the general form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{1}$$

where $M(x,y) = F_x(x,y)$ and $N(x,y) = F_y(x,y)$ for some function $F(x,y)$. They are also commonly written

$$M(x,y)dx + N(x,y)dy = 0.$$

There are a few notes worth making:

- Exact differential equations are not generally linear. In other words, this is a method for solving first-order *nonlinear* differential equations.
- The general solution for an exact equation is the implicit form $F(x,y) = C$.
- Although this is a distinct class of differential equations, it will share many similarities with first-order linear differential equations. Importantly, we will discover that there is often (although not always!) an integration factor required to make a differential equation in the "exact" form. This integration factor will take a different form than that of first-order linear equations.

The question then becomes, if we have a general differential equation of the form (1), how do we know if it is exact? The answer comes to us from recognizing the equality of mixed-order partial derivatives. For a general twice differentiable function $F(x,y)$, we have

$$\begin{aligned} \frac{\partial^2}{\partial y \partial x} F(x,y) &= \frac{\partial^2}{\partial x \partial y} F(x,y) \\ \implies \quad \frac{\partial}{\partial y} F_x(x,y) &= \frac{\partial}{\partial x} F_y(x,y) \\ \implies \quad \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x}. \end{aligned}$$

It can be shown that this is a necessary and sufficient condition for exactness. This is an easy check, but it will not tell us how to find the general solution. For that, we consider an example.

Example 1: Show that the following differential equation is exact and use this observation to find the general solution:

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2 \right) dy = 0.$$

We have $M(x,y) = 4xy^{1/2}$ and $N(x,y) = \frac{x^2}{y^{1/2}} + 2$. The required condition for exactness is easy to check:

$$\frac{\partial M}{\partial y} = \frac{2x}{y^{1/2}} = \frac{\partial N}{\partial x}.$$

It follows that the equation is exact and, consequently, that there is a solution of the form $F(x,y) = C$. It remains to find the solution. How might we accomplish this?

The key is to notice that the differential equations give rise to the system of equations

$$\begin{aligned}\frac{\partial F}{\partial x} &= M(x,y) = 4xy^{1/2}\\ \frac{\partial F}{\partial y} &= N(x,y) = \frac{x^2}{y^{1/2}} + 2.\end{aligned}$$

This can be solved by integrating either expression by the respective variable of the partial derivative. The first expression gives

$$F(x,y) = 2x^2y^{1/2} + g(y)$$

where we have to include an arbitrary function of y (i.e. the $g(y)$) because partial differentiation with respect to x would eliminate such a term. We now solve for $g(y)$ by taking the derivative of F with respect to the *other* variable, y. We have

$$\frac{\partial F}{\partial y} = \frac{x^2}{y^{1/2}} + g'(y).$$

We can see by comparing this equation with the previous system that we need to have $g'(y) = 2$. It follows that $g(y) = 2y + C$ so that the general solution is

$$2x^2y^{1/2} + 2y = C.$$

It is worth making a few notes on this process:

- It is important to remember that integrating a partial derivative requires us to add an additional term *of the other variable.*
- It is a general property that the solution will only be represented in implicit form. In other words, do not worry too much about solving for y in the final steps.

Now consider being asked to solve the differential equation

$$(4xy)dx + (x^2 + 2y^{1/2})dy = 0.$$

We notice immediately that this is just the previous example multiplied through by $y^{1/2}$. We suspect that this equation has the same solutions, and the same methods will apply, but we can see that

$$\frac{\partial M}{\partial y} = 4x \neq 2x = \frac{\partial N}{\partial x}.$$

In other words, the equation is no longer exact! This is a problem. We only know how to solve equations of this form if they are exact. We seem to be stuck.

The resolution comes by recognizing where the difference between the two equations came. We can change this expression into an exact form by dividing through by $y^{1/2}$ (or multiplying through by $y^{-1/2}$, if you prefer). It should be clear then that—just as with first-order linear equations—sometimes we will need to multiply through by some factor (also called an **integrating factor**!) in order to get the equation in the form we can use.

We might wonder if *all* equations of the form (1) can be made exact by multiplication by an integration factor. This was what happened for first-order linear differential equations, so it is not an unfair question. The answer in this case, however, is unfortunately a pronounced **NO**. There are many differential equations of the form (1) which cannot be manipulated so that they are exact. The question then becomes, which differential equations can be? Are there are conditions which guarantee a differential equation of the form (1) can be made exact by multiplication by an appropriate integration factor? And, if so, what is that integration factor?

The answer to these last questions is fortunately a **YES**. We have the following conditions and associated integration factors:

Proposition 5.1. *Consider a general differential equation of the form (1). Then:*

1. *If* $R(x) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N$ *is a function of* x *alone, then the integration factor*

$$\mu(x) = e^{\int R(x)\,dx}$$

will make (1) exact.

2. *If* $R(y) = \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)/M$ *is a function of* y *alone, then the integration factor*

$$\mu(y) = e^{\int R(y)\, dy}$$

will make (1) exact.

We will not justify these forms (although it is a good exercise!). Let's consider how they work for our specific example.

We need to check one or the other of the above conditions. We have

$$\frac{\partial M}{\partial y} = 4x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x.$$

To check whether the first condition is satisfied, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{4x - 2x}{x^2 + 2y^{1/2}}\right) = \left(\frac{2x}{x^2 + 2y^{1/2}}\right).$$

Since this is not a function of x alone, the first condition fails and we are not allowed to construct an integration factor out depending on x.

Now consider the second condition. We have

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{2x - 4x}{4xy}\right) = -\frac{2x}{4xy} = -\frac{1}{2y}.$$

Since this is a function of y alone, we are allow to construct an integration factor out of it. Setting $R(y) = -1/(2y)$, we have

$$\mu(y) = e^{\int R(y)\, dy} = e^{-\int \frac{1}{2y}\, dy} = e^{-\frac{1}{2}\ln(y)} = y^{-1/2}.$$

This is exactly integration factor we expected! Multiplying through the expression by $\mu(x) = y^{-1/2}$ gives

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2\right) dy = 0.$$

This is the earlier expression, which we have already shown in exact, and for which we already know the solution! The only trick was determining an appropriate integration factor. It took a little more work than in the case of linear first-order differential equations, but nevertheless we were able to accomplish the task.

There are a few notes worth making:

- We may (once again) exclude constants and absolute values in the integration required to determine the form of the integration factor.

- It will be very important to keep the conditions on the variables x and y straight (though practice!). The key terms are M_y and N_x, so that the coefficient of dx has a y derivative taken, and the coefficient of dy has an x derivative taken. If the wrong derivatives are evaluated, the methods will not work.

Example: Determine the solution of

$$y\cos(x)dx + (1-y^2)\sin(x)dy = 0.$$

Solution: We might notice that this equation is separable, but ignoring that for the time-being, we will treat as an exact (or nearly exact) equation. To check for exactness, we compute

$$M_y = \cos(x) \neq (1-y^2)\cos(x) = N_x.$$

So that differential equation is not exact. In order to check for an integration factor, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{\cos(x) - (1-y^2)\cos(x)}{(1-y^2)\sin(x)}\right) = \left(\frac{y^2\cos(x)}{(1-y^2)\sin(x)}\right).$$

This is clearly not a function of x alone, so we may remove it from consideration. The other condition gives

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{(1-y^2)\cos(x) - \cos(x)}{y\cos(x)}\right)\left(\frac{-y^2\cos(x)}{y\cos(x)}\right) = -y.$$

Since this is a function of y alone, we set $R(y) = -y$ and evaluate the integration factor

$$\mu(y) = e^{\int R(y)\,dy} = e^{-\int y\,dy} = e^{-\frac{y^2}{2}}.$$

We now multiply the expression through by this term. We have

$$ye^{-\frac{y^2}{2}}\cos(x)dx + (1-y^2)e^{-\frac{y^2}{2}}\sin(x)dy = 0.$$

This gives the system of necessary equations

$$\frac{\partial F}{\partial x} = M(x,y) = ye^{-\frac{y^2}{2}}\cos(x)$$
$$\frac{\partial F}{\partial y} = N(x,y) = (1-y^2)e^{-\frac{y^2}{2}}\sin(x).$$

The obvious choice (I hope!) is to integrate the first expression with respect to x. We have

$$F(x,y) = \int \frac{\partial F}{\partial x}\, dx = ye^{-\frac{y^2}{2}}\sin(x) + g(y).$$

Taking the derivative of this with respect to y yields

$$\frac{\partial F}{\partial y} = e^{-\frac{y^2}{2}}\sin(x) - y^2e^{-\frac{y^2}{2}}\sin(x) + g'(y) = (1-y^2)e^{\frac{y^2}{2}}\sin(x) + g'(y).$$

Comparing this with the second equation gives $g'(y) = 0$ so that $g(y) = C$. This gives the general (implicit) solution

$$F(x,y) = ye^{-\frac{y^2}{2}}\sin(x) = C.$$