

MATH 319, WEEK 5 & 6:

Second-Order Differential Equations

1 Second-Order Differential Equations

So far, we have only dealt with *first-order differential equations*. That is to say, we have only dealt with equations of the form

$$\frac{dy}{dx} = f(x, y). \quad (1)$$

We have developed a number of methods for solving this type of equations and, also, methods for analyzing the solutions when they cannot be found explicitly. We have seen a number of examples where equations of this form arise: population growth models, stirred-tank models, velocity/acceleration problems, etc. All in all, we should also feel pretty confident about our mastery of first-order differential equations.

It should come as no surprise, however, that the mathematical models of many physical phenomena *cannot* be represented by equations of the form (1). The simplest possible extension we can make it to consider *second-order* differential equations, i.e. equations of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \quad (2)$$

At first glance, equations (1) and (2) do not seem profoundly different, and so we might suspect that the methods we have learned for (1) will apply here. After all, the derivatives are still with respect to a single variable, and we can certainly verify solutions for (2) as easily as for (1)—we just plug them in! That is, however, right around where the similarities end. In fact, none of the methods we learned for first-order differential equations will work for second-order (or higher) ones. We will not be able to construct direction fields, separate variables, or use simple substitutions (except in very special cases!).

Before investigating the methods which will be applicable to solving second-order differential equations, we first motivate how such equations commonly arise in practice.

2 Motivation: Damped Spring / Pendulum

Consider the forces acting on a pendulum (or on an elongated spring). Suppose the rest position is $x = 0$, anything to the right of that is $x > 0$, and anything to the left is $x < 0$. If we move the pendulum to the right ($x > 0$), gravity acts against the pendulum to force it left ($F < 0$); conversely, if we move the pendulum to the left ($x < 0$), gravity acts against the pendulum to force it right ($F > 0$). (See Figure 1.)

If we consider a frictional force in addition to this “restoring force”, we have a similar interpretation except in terms of the *velocity*. If we imagine $v = 0$ as no velocity, $v > 0$ as movement to the right, and $v < 0$ as movement to the left, we have that friction always acts *against* the pendulum (i.e. $F < 0$ if $v > 0$ and $F > 0$ if $v < 0$).

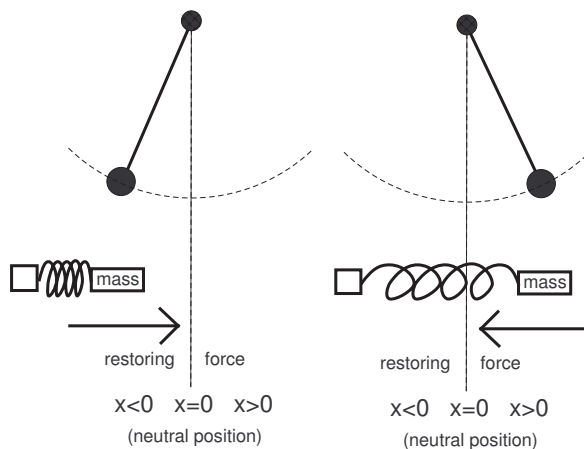


Figure 1: Restoring forces acting on a simple pendulum or a mass-spring. The force acts to restore the mass to its resting or neutral position.

Now let’s attempt to capture these forces more precisely. We will assume the following:

1. **Restoring force proportional to position:** That is to say, we will assume that $F_{restoring} = -kx$ for some $k > 0$. This satisfies our previous intuition ($F < 0$ for $x > 0$ and $F > 0$ for $x < 0$) although it is an approximation which does not hold for *high-amplitude* oscillating pendulums (i.e. pendulums that swing very far from the rest position).

2. **Frictional force proportional to velocity:** That is to say, we will assume that $F_{friction} = -cv = -c\frac{dx}{dt}$ for some $c > 0$. This again satisfies our previous intuition. It also makes sense that the more we increase our velocity, the more “drag” we will experience.

The question then becomes how to incorporate this into a differential equation model. The answer comes from Newton’s second law $F = ma$ (i.e. force equals mass times acceleration). We have

$$ma = m\frac{d^2x}{dt^2}$$

$$F = F_{restoring} + F_{friction} = -kx - c\frac{dx}{dt}.$$

Putting this all together gives us the combined differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \quad (3)$$

(Notice that we could derive the same differential equations, with a slightly different interpretation for the constants involved, by considering a mass-spring example obeying Hooke’s law.)

This differential equation may not look like much, but it will be our canonical example (plus or minus a few modifications) for the next few weeks. There are a few important things to notice about it:

- It is a *second-order* differential equation. Furthermore, it should not take much convincing that the techniques we learned in the early portion of this course (e.g. separating variables, finding integrating factors) are not going to work for finding a solution of such equations (or higher-order equations).
- It is *linear* and has *constant coefficients*. In some senses, this is the best possible case, and we will always been able to find solutions. A little later on, we will deal with differential equations like this with a *forcing* term on the right-hand side, i.e. equations like

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t).$$

Such equations are called *non-homogeneous* while equations of the form (3) are called *homogeneous*. (Note the difference in meaning between homogeneous *first-order* differential equations!) We will also

consider linear second-order differential equations which do not have constant coefficients, i.e. differential equations of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad \text{and} \quad \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x).$$

- There is a further subtlety regarding initial conditions of second-order differential equations. Consider looking at a snapshot of a pendulum extended to the right and asking the question of what happened to the pendulum in the next moments after the snapshot was taken. We should quickly realize that there are three cases:
 1. If the pendulum was *at rest*, it will slowly pick up speed from rest and move toward its resting position.
 2. If the pendulum was *swinging to the right*, it will continue to the right, lose speed, and eventually reverse (or swing over the top).
 3. If the pendulum was *returning from the right*, it is already moving and will quickly return to the rest position (and probably overshoot it).

In any case, we see that it is very important to consider not only the *position* of the pendulum at the time the snapshot was taken, but also the *velocity*. In general, for second-order differential equations, we will always need *two* initial conditions.

(Note: This should also be intuitive in mathematical principle. To resolve a second-order differential equations, if we were just able to integrate, we would have to integrate *twice* and so end up with *two* integration constants. We would correspondingly need two pieces of information to solve uniquely for both constants.)

3 Second-Order Linear Differential Equations with Constant Coefficients

Consider the general homogeneous second-order differential equation with constant coefficients given by

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{4}$$

How might we go about finding a solution for such an equation? We cannot separate the variables, or find an integrating factor, or find an obvious substitution which will reduce the differential equation to first-order. So what is there left to do?

The answer is perhaps unsatisfying—especially given how often we have avoided doing this so far in the course—but we are actually going to *guess*. At the very least we will take an educated guess. We know one function which behaves particularly well under the operation of differentiation: the exponential function e^{rx} . We also know that this is the solution to the first-order linear homogeneous equation with constant coefficients, i.e. the differential equation

$$\frac{dy}{dx} = ry.$$

So we will guess that a solution of (4) has the form

$$y(x) = e^{rx}$$

for some r and see what happens. In the worst case scenario, even if this does not work out, we have not lost a great deal of time. It is easy to take derivatives of the exponential function!

Example: Find a solution of the following second-order differential equation in the form $y(x) = e^{rx}$:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0. \tag{5}$$

Solution: We will guess that the solution has the form $y(x) = e^{rx}$. This gives

$$y = e^{rx}, \quad \frac{dy}{dx} = re^{rx}, \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

so that

$$\begin{aligned} \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y &= r^2e^{rx} - 5re^{rx} + 4e^{rx} \\ &= e^{rx}(r^2 - 5r + 4) \\ &= e^{rx}(r - 1)(r - 4). \end{aligned}$$

The only way for this to equal zero is to have $r = 1$ or $r = 4$. It follows that either

$$y_1(x) = e^x, \quad \text{and} \quad y_2(x) = e^{4x}$$

are solutions of the differential equation.

(Note: If we are unconvinced at this point, we could just check directly. We have that $y_1(x) = e^x$ gives

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^x - 5e^x + 4e^x = 0$$

and $y_2(x) = e^{4x}$ gives

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = (16e^{4x}) - 5(4e^{4x}) + 4e^{4x} = 0.)$$

We should pause to make a few notes at this point:

- We have seen before that differential equations can have multiple answers. Previously, however, we were able to resolve this by introducing an initial condition. In this case, however, we have two completely different *forms* of solutions. It is easy to check that any functions of the form

$$y_1(x) = C_1e^x \quad \text{and} \quad y_2(x) = C_2e^{4x}$$

are solutions. In fact, any function of the form

$$y(x) = C_1y_1(x) + C_2y_2(x) = C_1e^x + C_2e^{4x}$$

will be a solution, where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. This is called the **general solution** of the differential equation.

- In order to find the **particular solution**—i.e. in order to solve for C_1 and C_2 —we will need to introduce *two* initial conditions. For instance, if we have the initial information $y(0) = 1$ and $y'(0) = 0$ we can compute that

$$y'(x) = C_1e^x + 4C_2e^{4x}$$

so that

$$\begin{aligned} y(0) = 1 &\implies C_1e^{(0)} + C_2e^{4(0)} = C_1 + C_2 = 1 \\ y'(0) = 0 &\implies C_1e^{(0)} + 4C_2e^{4(0)} = C_1 + 4C_2 = 0. \end{aligned}$$

We can solve this system of two variables in two unknowns by any method we happen to know. (Matrix algebra is helpful but not necessary.) The second equation gives us

$$C_1 = -4C_2$$

so that the first equation reduces to

$$C_1 + C_2 = (-4C_2) + C_2 = -3C_2 = 1.$$

It follows that $C_2 = -1/3$ and $C_1 = 4/3$ so that relevant particular solution is

$$y(x) = \frac{4}{3}e^x - \frac{1}{3}e^{4x}.$$

3.1 Properties of Solutions

We should probably step back at this point and consider some general properties of solutions of second-order differential equations. The first property, which we have already hinted at, is that solutions may be *combined* to former bigger solutions.

Theorem 3.1 (Principle of Superposition). *Consider the general homogeneous second-order differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Suppose that $y_1(x)$ and $y_2(x)$ are solutions of the equation. Then

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

is a solution.

Proof. As with any formal mathematical proof, it may help to first write out our given information and the objective—i.e. the thing we are trying to prove—in mathematical terms. In this case, we want to use the fact that $y_1(x)$ and $y_2(x)$ are solutions to show $y(x) = C_1y_1(x) + C_2y_2(x)$ is a solution. The given information is that $y_1(x)$ and $y_2(x)$ are solutions, which means that

$$\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 = 0. \quad (6)$$

We will have to use this at some point. The objective is to show that $y(x) = C_1y_1(x) + C_2y_2(x)$ is a solution. Let's check! On the left-hand side,

we have

$$\begin{aligned}
& \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y \\
&= \frac{d^2}{dx^2} [C_1y_1(x) + C_2y_2(x)] + p(x)\frac{d}{dx} [C_1y_1(x) + C_2y_2(x)] + q(x) [C_1y_1(x) + C_2y_2(x)] \\
&= C_1\frac{d^2y_1}{dx^2} + C_2\frac{d^2y_2}{dx^2} + C_1p(x)\frac{dy_1}{dx} + C_2p(x)\frac{dy_2}{dx} + C_1q(x)y_1 + C_2q(x)y_2 \\
&= C_1 \left[\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1 \right] + C_2 \left[\frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 \right].
\end{aligned}$$

It is easy to lose ourselves in this algebra and lose track of where we were trying to get to. The important thing is that we have our given information written out clearly. Referring back to (6), we immediately see that the left-hand side simplifies to zero, since each bracketed term is exactly in the form of the given information. Tracing back to the original equation, we have that

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

But this is exactly what it means for $y(x)$ to be a solution of the differential equation! In other words, we accomplished exactly what we set out to show. We are done. \square

Let's stop to make a few notes:

- The form $y(x) = C_1y_1(x) + C_2y_2(x)$ is called a *linear combination* of the solutions $y_1(x)$ and $y_2(x)$. The reason for the terminology should be obvious to those with some background in linear algebra (think linear combination of vectors!) but is not important enough for the purposes of this course to dwell on here.
- In order to apply the principle of superposition, it was very important that the equation be linear and that the right-hand side was zero, i.e. it was a homogeneous equation. It is very easy to construct examples where the principle fails for non-linear differential equations or differential equations with a non-trivial right-hand side. For example, the nonlinear differential equation

$$y'(x) - y^{1/2} = 0$$

has the general solution

$$y(x) = \frac{(x - C)^2}{4}.$$

It can be seen, however, that we may not take even a trivial linear combination (i.e. just scaling!) for this function while maintaining the property of being a solution. For instance, the function

$$y_1(x) = 4y(x) = (x - C)^2$$

fails to be a solution because $y'(x) = 2(x - C)$ and $y^{1/2} = x - C$ (for $x \geq C$). It is also necessary for the equation to be *homogeneous*. For instance, the differential equation

$$y'(x) - y = e^x$$

has the general solution

$$y(x) = (x + C)e^x.$$

It can be easily checked that $y(x) = xe^x$ satisfies the differential equation; however, the function $y_1(x) = 2y(x) = 2xe^x$ does not.

The principle of superposition allows us to *build* solutions. It says that, given any two solutions, we can make another solution by adding them together. It is also clear that this can be extended to as many solution as we like—that is to say, we could bring three solutions together, or four, or five, and build bigger and bigger solutions. While this tells us something very important about solutions, it also raises a few very important questions:

1. Is there a fundamental set of *building blocks* with which we can build *every* solution of the differential equation?
2. How many are there?
3. How do we tell them apart?

These are very important questions! We know that $y(x) = C_1e^x + C_2e^{4x}$ is a solution of our example differential equation, but are e^x and e^{4x} the only possible choices? Could there be a third function that we just have not found? How do we know we have found *everything* from which a solution could be built?

To answer this question, we introduce the following very important term, known as the Wronskian.

Definition 3.1. The **Wronskian** of two functions $y_1(x)$ and $y_2(x)$ is the term

$$W(y_1, y_2, x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Note: For those of you who have taken some linear algebra, the Wronskian may be more recognizable in the determinant form

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

This form generalizes easily to higher dimensions, but that will not factor significantly in this course.

Although it is not obvious, the Wronskian plays a significant role in whether an initial value problem for a second-order differential equation is well-posed. We have the following result.

Theorem 3.2 (Theorem 3.2.4 in text). *Suppose $y_1(x)$ and $y_2(x)$ are solutions of the second-order homogeneous differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Then every solution of the differential equation can be expressed in the form

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

for some $C_1, C_2 \in \mathbb{R}$ if and only if the Wronskian $W(y_1, y_2, x)$ is non-zero for some value of x .

It may be hard at first glance to understand exactly what this is saying, and why it is important. Perhaps this is best fielded by continuing our example. We previously found that $y_1(x) = e^x$ and $y_2(x) = e^{4x}$ were solutions of the given differential equation. We were then able to convince ourselves (through either inspection, or Theorem 3.1) that any function of the form $y(x) = C_1e^x + C_2e^{4x}$ was also a solution. We could *not*, however, be certain that every solution had this form. In order to do that, we need to check the Wronskian. We have that

$$y_1'(x) = e^x \quad \text{and} \quad y_2'(x) = 4e^{4x}$$

so that

$$\begin{aligned} W(y_1, y_2, x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= (e^x)(4e^{4x}) - (e^x)(e^{4x}) \\ &= 3e^{5x}. \end{aligned}$$

Since this is non-zero everywhere, we can finally say that the solution form $y(x) = C_1e^x + C_2e^{4x}$ is complete. That is to say, we are guarantee that we

have not missed any solutions! Every solution can indeed be written as a linear combination of e^x and e^{4x} .

Note: It is natural to ask which kinds of solutions $y_1(x)$ and $y_2(x)$ do not satisfy the Wronskian condition in Theorem 3.2. That is to say, if we have two solutions, how likely is it that we cannot construct the whole solution class out of them? The answer is a very definitive *not often*—in fact, they have to be multiples of one another, i.e. $y_2(x) = Cy_1(x)$ for some C . That means that almost all pairs of elementary functions (e.g. e^x , e^{-2x} , $\ln(x)$, x^2 , $\sin(x)$, etc.) have a non-zero Wronskian. We will nevertheless see that the Wronskian can be used in very powerful ways to *construct* solutions of differential equations.

This tells us what the solution set looks like—a linear combination of basic solutions—but it does not tell us how to *find* these basic solutions. In fact, this can be *very hard* for general linear differential equations with variable coefficients! There are many examples, even second-order examples, where the solution may only be expressed as a power series approximation. We will consider this later in the course. For now, we will return to consideration of second-order linear homogeneous differential equations with constant coefficients.

4 Solution Forms (Constant Coefficients)

Reconsider the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \quad (7)$$

In order to solve this equation, we guess the general form $y(x) = e^{rx}$. We can see very quickly that this yields

$$ar^2e^{rx} + bre^{rx} + e^{rx} = e^{rx}(ar^2 + br + c) = 0.$$

Since $e^{rx} > 0$ for all $x \in \mathbb{R}$, it follows that we must have $ar^2 + br + c = 0$ in order to have a solution. It follows by the quadratic formula that we have

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (8)$$

We observe that there are three possible cases:

1. If $b^2 - 4ac > 0$ we will have two distinct real values r_1 and r_2 .
2. If $b^2 - 4ac = 0$ we will have one repeated real value r .
3. If $b^2 - 4ac < 0$ we will have a complex conjugate pair $r = \alpha \pm \beta i$, where $\alpha = \text{Re}(r)$ and $\beta = \text{Im}(r)$.

We have already seen what happens for the first case. The second two cases are the trickier cases. They are captured by the following result.

Theorem 4.1. *Consider the second-order linear homogeneous differential equation (7) with constant coefficients. Let r_1 and r_2 be defined by (8). Then the general solution of (7) is:*

1. If $b^2 - 4ac > 0$ the general solution is $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
2. If $b^2 - 4ac = 0$ the general solution is $y(x) = C_1 e^{rx} + C_2 x e^{rx}$.
3. If $b^2 - 4ac < 0$ the general solution is

$$y(x) = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

Proof. We know that we only need to find two linearly independent solutions. We have the following cases:

1. Case 1: If the guess $y(x) = e^{rx}$ produces two distinct real values r_1 and r_2 , we have that $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ essentially for free. The only thing remaining is to show that the Wronskian is non-zero when $r_1 \neq r_2$ (Check!). Finally, by Theorem 3.2, we have

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

2. Case 2: If the guess $y(x) = e^{rx}$ only produces the single solution $y_1(x) = e^{rx}$ then we need to find another solution by another method. Recall that the measure of solutions being different was that the Wronskian was non-zero. It should not take much convincing (check!) to believe that the only way two solutions can have a zero Wronskian is if they are multiples of one another. For instance, if $y_1(x) = e^{2x}$ is a solution, and proposed $y_2(x) = 5e^{2x}$ as a second basic solution, we would quickly find that $W(y_1, y_2, x) = 0$. That means these solutions are not sufficient different to construct the general solution set $y(x) = C_1 y_1(x) + C_2 y_2(x)$ out of them. We cannot have solutions of the form $y_2(x) = C y_1(x)$!

What we require, therefore, is that there is another another function $u(x)$ (which is not a constant!) so that

$$y_2(x) = u(x)y_1(x). \quad (9)$$

This quantifies the fact that $y_2(x)$ and $y_1(x)$ must have more variance between them than just constant multiplication. We want to determine a non-trivial function $u(x)$ for which $y_2(x)$ is a solution of (7) given that $y_1(x) = e^{rx}$ is a solution.

We know that $y_1(x) = e^{rx}$ is a solution. This means that

$$ay_1'' + by_1' + cy_1 = e^{rx}(ar^2 + br + c) = 0 \quad (10)$$

and, since $b^2 - 4ac = 0$, we have that

$$r = -b/(2a) \quad (\text{alternatively } 2ar + b = 0) \quad (11)$$

It follows from (9) that

$$\begin{aligned} y_2(x) &= u(x)y_1(x) = u(x)e^{rx} \\ y_2'(x) &= u'(x)e^{rx} + ru(x)e^{rx} \\ y_2''(x) &= u''(x)e^{rx} + 2ru'(x)e^{rx} + r^2u(x)e^{rx}. \end{aligned}$$

Substituting this into the left-hand side of the differential equation gives

$$\begin{aligned} ay_2''(x) + by_2'(x) + cy_2(x) &= a(u''(x)e^{rx} + 2ru'(x)e^{rx} + r^2u(x)e^{rx}) \\ &\quad + b(u'(x)e^{rx} + ru(x)e^{rx}) + cu(x)e^{rx} \\ &= u(x)e^{rx}(ar^2 + br + c) + u'(x)e^{rx}(2ar + b) + au''(x)e^{rx} \end{aligned}$$

Here is where we may final use our given information. The term corresponding to $u(x)$ must be zero because $y_1(x) = e^{rx}$ is a solution (10), while the term corresponding to $u'(x)$ must be zero because r is a repeated root (11). Since the right-hand side of the equation is zero, we have that, in order for $y_2(x)$ to be a solution, it is enough to have $au''(x)e^{rx} = 0$. The only way this can happen is if

$$u''(x) = 0 \implies u(x) = Ax + B.$$

It follows that the solution $y_2(x)$ is given by

$$y_2(x) = u(x)y_1(x) = (Ax + B)e^{rx} = Axe^{rx} + Be^{rx}.$$

We now have

$$\begin{aligned} y(x) &= C_1 y_1(x) + C_2 y_2(x) = C_1 e^{rx} + C_2 (A x e^{rx} + B e^{rx}) \\ &= (C_1 + C_2 B) e^{rx} + C_2 A x e^{rx} = \tilde{C}_1 e^{rx} + \tilde{C}_2 x e^{rx}. \end{aligned}$$

Since it can be easily checked the the Wronskian of e^{rx} and $x e^{rx}$ is non-zero for all $x \in \mathbb{R}$, we are done.

3. Case 3: If the guess $y(x) = e^{rx}$ yields a complex conjugate pair, we have that that $r_{1,2} = \alpha \pm \beta i$ where $\alpha = \text{Re}(r)$ and $\beta = \text{Im}(r)$ so that

$$y_{1,2}(x) = e^{(\alpha \pm \beta i)x} = e^{\alpha x} e^{\pm \beta i x}.$$

This, however, involves the imaginary number $i = \sqrt{-1}$, while we are clearly only interested in real-valued solutions. It turns out that we can use some arithmetic to get rid of the imaginary parts of the equation and find two linearly independent *real-valued* solutions to (7).

It is a well-known fact of complex analysis that

$$e^{ix} = \cos(x) + i \sin(x).$$

This formula is known as **Euler's formula** (not related to the Euler method in numerical methods!). It is certainly not an obvious formula by any means, but it can be verified by taking the Taylor series expansions of the left- and right-hand sides of the equation. At any rate, it now follows that we have the solutions

$$y_1(x) = e^{\alpha x} e^{\beta i x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x))$$

and

$$y_2(x) = e^{\alpha x} e^{-\beta i x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x))$$

We know that *any linear combination* of these functions produces a solution of (7). In particular, if we can find a linear combination of these solutions which are *real-valued* then we can forget about this whole complex valued mess we have gotten ourselves into.

In fact, we can do just that! After trying for a little while, we might notice that

$$\tilde{y}_1(x) = \frac{1}{2} y_1(x) + \frac{1}{2} y_2(x) = e^{\alpha x} \cos(\beta x).$$

That is to say, by taking this linear combination, we can eliminate all dependence on the complex value i . This is one solution, but we

expect that there are actually two. So we must find another way to eliminate the dependence on i .

The method of doing so is not obvious at all, but it is easy to verify that taking the *complex* linear combination

$$\tilde{y}_2(x) = -\frac{i}{2}y_1(x) + \frac{i}{2}y_2(x) = e^{\alpha x} \sin(\beta x)$$

we obtain a second solution which has a non-trivial Wronskian on $x \in \mathbb{R}$ (Check!). That's all we need to do! The general solution is

$$\begin{aligned} y(x) &= C_1\tilde{y}_1(x) + C_2\tilde{y}_2(x) \\ &= C_1e^{\alpha x} \cos(\beta x) + C_2e^{\alpha x} \sin(\beta x). \end{aligned}$$

□

Example 1: Find the general solution of $4y''(x) + 12y'(x) + 9y(x) = 0$. Then find the particular solution for $y(0) = 2$ and $y'(0) = 0$.

Solution: We guess the solution form $y(x) = e^{rx}$. This gives

$$4y''(x) + 12y'(x) + 9y(x) = e^{rx}(4r^2 + 12r + 9) = e^{rx}(2r + 3)^2 = 0.$$

It follows that we only have a solution if $r = -3/2$. Since this is a repeated root, we are in Case 2 and the general solution is given by

$$y(x) = C_1e^{-(3/2)x} + C_2xe^{-(3/2)x}.$$

To solve for the particular solution, we compute

$$y'(x) = -\frac{3}{2}C_1e^{-(3/2)x} + C_2e^{-(3/2)x} - \frac{3}{2}C_2xe^{-(3/2)x}.$$

The conditions $y(0) = 2$ and $y'(0) = 0$ gives the system

$$\begin{aligned} C_1 &= 2 \\ -\frac{3}{2}C_1 + C_2 &= 0. \end{aligned}$$

We can quickly solve this to get $C_1 = 2$ and $C_2 = 3$. It follows that the particular solution is

$$y(x) = 2e^{-(3/2)x} + 3xe^{-(3/2)x}.$$

Example 2: Find the general solution of $y''(x) + 2y'(x) + 2y(x) = 0$. Then find the particular solution for $y(0) = 1$ and $y'(0) = -1$.

Solution: We guess the solution $y(x) = e^{rx}$. This gives

$$y''(x) + 2y'(x) + 2y(x) = e^{rx}(r^2 + 2r + 2) = 0.$$

The quadratic formula gives the solution

$$r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i.$$

Since this a complex root, we are in case 3 and the general solution is

$$y(x) = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x).$$

To solve for the particular solution, we compute

$$\begin{aligned} y'(x) &= -C_1 e^{-x} \cos(x) - C_2 e^{-x} \sin(x) - C_1 e^{-x} \sin(x) + C_2 e^{-x} \cos(x) \\ &= -C_1 e^{-x} (\cos(x) + \sin(x)) + C_2 e^{-x} (\cos(x) - \sin(x)). \end{aligned}$$

The conditions $y(0) = 1$ and $y'(0) = -1$ gives the system

$$\begin{aligned} C_1 &= 1 \\ -C_1 + C_2 &= -1. \end{aligned}$$

It follows immediately that $C_1 = 1$ and $C_2 = 0$ so that the particular solution is

$$y(x) = e^{-x} \cos(x).$$