MATH 319, WEEK 7:
Non-Homogeneous Linear DEs

1 Nonhomogeneous Linear Differential Equations

Suppose we want to solve the second-order differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = 12x. \tag{1}$$

The only difference between this and the type of equations we have been considering so far is the term $12x$ on the right-hand side. This extra term is enough to make the equation *non-homogeneous*.

We might think at first glance that the technique used to solve *homogeneous* equations might work. That is to say, we guess that the solution has the exponential form $y(x) = e^{rx}$. This gives

$$\frac{d^2y}{dx^2} + 4y(x) = e^{rx}\left(r^2 + 4\right) = 12x.$$

It should not take much convincing that there is no value of r which satisfies this equation for all x. The guess we used for homogeneous linear equations will not work for non-homogeneous equations. So, what can we do?

The answer may be unsatisfying, but it should not be surprising. We are just going to *guess something else*. This time, however, we are going to have to guess a different solution form—in particular, we are going to have to guess a function which, when substituted in the left-hand side, gives the non-homogeneous term on the right-hand side. This is surprisingly easy to do! We can see immediately that $y(x) = 3x$ is a solution of the equation since it satisfies $4y(x) = 12x$ and $y''(x) = 0$. (Probably the only way we would not have seen this would have been to overthink the problem!)

We will say that we have found a *particular* solution $y_p(x) = 3x$ since it satisfies the differential equation. We might wonder, however, how to build a general solution out of this observation. It is not as easy as multiplying the found solution by a constant, since $y(x) = Cx$ is clearly not a solution for all $C \in \mathbb{R}$. We must be a little more careful. After some thought, we might realize that if we could find a solution to the *homogeneous* equation

$$\frac{d^2y}{dx^2} + 4y(x) = 0 \tag{2}$$

then we could add *that solution* to the particular solution $y_p(x) = 3x$ and still satisfy (1). The reason should be clear:

1. Plugging $y_p(x)$ in the left-hand side of (1), we obtain $12x$.
2. Plugging the solution of (2) in the left-hand side of (1), we obtain 0.
3. Consequently, plugging both in at the same time produces $12x + (0) = 12x$, so that it is still a solution of (1).

Since we know that $y_c(x) = C_1 \sin(2x) + C_2 \cos(2x)$ is the general solution of (2), we may quickly reason that

$$y(x) = y_c(x) + y_p(x) = C_1 \sin(2x) + C_2 \cos(2x) + 3x$$

is the general solution of (1).

We have actually stumbled upon the general solution method for solving second-order (and higher) linear differential equations.

Theorem 1.1 (Theorem 3.5.2 in text)**.** *Consider a second-order linear, nonhomogeneous differential equation of the form*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = g(x). \tag{3}$$

Then any solution of (3) can be written

$$y(x) = y_c(x) + y_p(x)$$

where $y_p(x)$ *is any* particular solution *of (3) and the* complementary function $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ *is the general solution of the homogeneous system*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0. \tag{4}$$

Note: Since we already know how to solve homogeneous second-order linear DEs with constant coefficients, this result tells us that we need only worry about finding $y_p(x)$! In general, however, it may be difficult to determine the complementary solution $y_c(x)$ *if* the coefficients are allowed to vary with time.

2 Method of Undetermined Coefficients

The question then becomes how we find the particular solution $y_p(x)$. We will see two methods in this class. The first will be an extension of what we have seen before: we will just guess! The second method, called *variation of parameters*, will not require any guessing but will require a significantly greater amount of work (and some potentially tricky integration). The method presented first, called *the method of undetermined coefficients*, therefore is the preferred method whenever it can be applied.

We notice that the left-hand side of (3) involves just y and its derivatives. What we need is a form of $y_p(x)$ which can be differentiated to give the form of $g(x)$ on the right-hand side. We notice that

$$\begin{aligned}\frac{d}{dx}[\text{polynomial}] &= \text{polynomial}\\ \frac{d}{dx}[\text{exponential}] &= \text{exponential}\\ \frac{d}{dx}[\text{sine and/or cosine}] &= \text{sine and/or cosine.}\end{aligned}$$

This suggests that we guess trial functions $y_p(x)$ of the following forms:

$$\begin{aligned}y_p(x) &= A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\\ y_p(x) &= Be^{rt}\\ y_p(x) &= A\cos(ax) + B\sin(ax)\end{aligned} \tag{5}$$

where the various coefficients A_n, A_0, A, B, etc. are undetermined constants. In order to solve for the constants in the trial function $y_p(x)$, we will need to plug the function into (3).

To summarize, we have the following steps for a linear homogeneous differential equations with constant coefficients (3):

1. Find the general solution $y_c(x)$ of the associated homogeneous equation (4).

2. Select a trial function $y_p(x)$ of some combination of the forms (5) (depending on $f(x)$).

3. Plug the trial function $y_p(x)$ into (3) and solve for the undetermined coefficients.

4. Write the general solution as $y(x) = y_c(x) + y_p(x)$.

Example 1: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = e^{-x} - 3x^3.$$

Solution: We need to first solve the homogeneous equation

$$\frac{d^2y}{dx^2} + 4y(x) = 0.$$

The guess $y_c(x) = e^{rx}$ gives $e^{rx}(r^2+4) = 0$ so that $r = \pm 2i$. It follows that

$$y_c(x) = C_1 \cos(2x) + C_2 \sin(2x).$$

We now need to use a trial function $y_p(x)$ with a suitable form that it could give $e^{-x} - 3x^3$ after differentiation. We try

$$\begin{aligned} & y_p(x) = Ae^{-x} + Bx^3 + Cx^2 + Dx + E \\ \Longrightarrow \quad & y_p'(x) = -Ae^{-x} + 3Bx^2 + 2Cx + D \\ \Longrightarrow \quad & y_p''(x) = Ae^{-x} + 6Bx + 2C. \end{aligned}$$

It follows that the differential equation gives

$$\begin{aligned} & y_p''(x) + 4y_p(x) \\ & = (Ae^{-x} + 6Bx + 2C) + 4(Ae^{-x} + Bx^3 + Cx^2 + Dx + E) \\ & = 5Ae^{-x} + 4Bx^3 + 4Cx^2 + (6B + 4D)x + (2C + 4E) \\ & = e^{-x} - 3x^3. \end{aligned}$$

It follows that we need to satisfy

$$\begin{aligned} 5A &= 1 \\ 4B &= -3 \\ 4C &= 0 \\ 6B + 4D &= 0 \\ 2C + 4E &= 0. \end{aligned}$$

It follows that we have $A = 1/5$, $B = -3/4$, $C = 0$, $D = 9/8$, and $E = 0$. The corresponding particular solution is

$$y_p(x) = \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

The general solution is therefore

$$y(x) = y_c(x) + y_p(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

Note: It is possible that we may need to use more complicated combinations of these functions. For instance, if the forcing term is $e^x \sin(x)$, we will need to use $y_p(x) = Ae^x \cos(x) + Be^x \sin(x)$. A term like $x^2 e^{-x}$ would need $(Ax^2 + Bx + C)e^{-x}$, and so on.

Note: The arguments *inside* the trigonometric and exponential terms are also important. If there are distinct constants, we will need to use distinct trial functions. For instance, the forcing term $f(x) = \sin(2x) + \cos(3x)$ requires the trial function $y_p(x) = A\cos(2x) + B\sin(2x) + C\cos(3x) + D\cos(3x)$.

3 Alternative Trial Forms

Now find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = \cos(2x).$$

We already have the complementary function $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$. We need to guess the form of the trial function $y_p(x)$. We need terms which can produce $\sin(2x)$ upon differentiation so we choose

$$y_p(x) = A\cos(2x) + B\sin(2x).$$

This gives

$$\begin{aligned} y_p'(x) &= -2A\sin(2x) + 2B\cos(2x) \\ y_p''(x) &= -4A\cos(2x) + 4B\sin(2x). \end{aligned}$$

It follows that we have

$$\begin{aligned} &y_p''(x) + 4y_p(x) \\ &= -4A\cos(2x) + 4B\sin(2x) + 4(A\cos(2x) + B\sin(2x)) = 0. \end{aligned}$$

We need to match constants so that this equals $f(x) = \sin(2x)$ but the term has vanished. We have nothing left to work with! Something has gone terribly wrong, but what?

We might notice that we should have expected this. After all, the complementary function is $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$, which meant we know that the combination of functions in the trial function *had to* vanish when it was substituted into the left-hand side of our differential equation. This raises a very important concern which we will have to identify:

- The trial functions (5) will only work if the individual functions *do not* appear in the complementary function $y_c(x)$.

In other words, if a forcing term coincides with a term already contained in the dynamics of the unforced system, we will not be able to construct nontrivial trial function in the same way as we have been.

It turns out that in this case we will have to use *different* trial functions. What we really need to do is generate *other* independent solutions (in the sense of having a non-trivial Wronskian). We have already done this! For instance, we found that if we had a solution $y_1(x) = e^x$ and needed another linearly independent one, that we could use $y_2(x) = xe^x$. If we need another one, we went up to $y_3(x) = x^2e^x$, and so on.

The same trick will work here! We will take our trial functions to be the same as used in (5) but with as many powers of x are required to give new functions. If the terms in the trial functions (5) appear in the complementary function $y_c(x)$, we must instead use the trial functions

$$\begin{aligned} y_p(x) &= A_n x^{n+s} + A_{n-1} x^{n+s-1} + \cdots + A_1 x^{s+1} + A_0 x^s \\ y_p(x) &= B x^s e^{rt} \\ y_p(x) &= A x^s \cos(ax) + B x^s \sin(ax) \end{aligned} \tag{6}$$

where s is the lowest power which produces a term which is independent of those in the complementary solution.

Example 1: Reconsider the example

$$\frac{d^2y}{dx^2} + 4y(x) = \cos(2x).$$

The complementary function was $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$ so we are not allowed to use $y_p(x) = A \cos(2x) + B \sin(2x)$ as a trial function. Instead, we must use

$$y_p(x) = Ax \cos(2x) + Bx \sin(2x).$$

This gives

$$y_p'(x) = A\cos(2x) + B\sin(2x) - 2Ax\sin(2x) + 2Bx\cos(2x)$$
$$y_p''(x) = 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x).$$

Plugging into the DE gives

$$\begin{aligned} & y_p'' + 4y_p \\ &= 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x) \\ &\quad + 4(Ax\cos(2x) + Bx\sin(2x)) \\ &= 4B\cos(2x) - 4A\sin(2x) \\ &= \cos(2x). \end{aligned}$$

It follows that we need $A = 0$ and $B = 1/4$ so that we have the particular solution

$$y_p(x) = \frac{1}{4}x\sin(2x).$$

The general solution of the differential equation is therefore

$$y(x) = C_1\cos(2x) + C_2\sin(2x) + \frac{1}{4}x\sin(2x).$$

While this seems like a simple fix, there are several subtleties which may arise. For instance, consider the following.

Example 2: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y(x) = xe^{-2x}.$$

Solution: We know now that we *always* need to determine the complementary solution $y_c(x)$ before shifting our focus to the particular solution $y_p(x)$. So we solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y(x) = 0$$

by substituting $y(x) = e^{rx}$ and get

$$e^{rx}(r^2 + 4r + 4) = e^{rx}(r + 2)^2 = 0$$

It follows that we have a repeated root $r = -2$, corresponding to the general solution

$$y_c(x) = C_1e^{-2x} + C_2xe^{-2x}.$$

We want to now find the particular solution $y_p(x)$. Our old guess, based on the right-hand side being xe^{-2x}, would be

$$y_p^{(1)}(x) = Axe^{-2x} + Be^{-2x}.$$

We recognize now, however, that the solution forms coincide with the fundamental solutions of the complementary solution $y_c(x)$ found above—e^{-2x} and xe^{-2x}. These forms *cannot* be a part of any particular solution, because we already know they evaluate to zero when substituted into the left-hand side of the differential equation.

Our first attempt at resolving this problem is to multiply the trial function form by x. This gives us the second candidate function

$$y_p^{(2)}(x) = Ax^2e^{-2x} + Bxe^{-2x}.$$

This is certainly better, but there is *still* a problem. The function x^2e^{-2x} is not contained in the complementary solution, but the function xe^{-2x} is. We cannot gain any information from this function, and x^2e^{-2x} is not enough to get the job done on its own. We saw that polynomial right-hand sides needed one more degree of freedom (unsolved coefficients) than their highest power. The same applies here. We need two constants to solve for, since the highest power term on the right-hand side is x, but only have one with the current trial function. This trial function will not succeed.

In fact, the correct trial function is obtained by multiplying by x^2. This gives

$$y_p(x) = Ax^3e^{-2x} + Bx^2e^{-2x}.$$

Neither of the forms here are contained in the complementary function so that we have two constants to solve for—exactly as many as demanded by the non-homogeneous term xe^{-2x} in the differential equation. We can now plug this into the differential equation. First, we compute

$$y_p'(x) = 2Bxe^{-2x} + (3A - 2B)x^2e^{-2x} - 2Ax^3e^{-2x}$$

and

$$y_p''(x) = 2Be^{-2x} + (6A - 8B)xe^{-2x} + (-12A + 4B)x^2e^{-2x} + 4Ax^3e^{-2x}.$$

The differential equation gives us

$$y_p''(x) + 4y_p'(x) + 4y_p(x) = 2Be^{-2x} + 6Axe^{-2x} = xe^{-2x}.$$

It follows that $B = 0$ and $A = 1/6$ so that the particular solution is

$$y_p(x) = \frac{1}{6}x^3 e^{-2x}.$$

The general solution is therefore

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{6}x^3 e^{-2x}.$$

This was certainly different than what we might have expected, so we should take a moment to clean up what we know so far.

1. If the functions in the naive trial function overlap with those in the complementary solution, we need to multiply the trial function by x ($s = 1$ in the earlier equation).

2. If the functions in the naive trial function overlap with those in the complementary solution *and* a root is repeated in the complementary function, we need to multiply the trial function by x^2 ($s = 2$ in the earlier equation).

For second-order equations, that is all that can happen, although this generalizes in the natural way to higher-order systems. Let's consider another subtlety which may arise to make sure we have not let anything slip between the cracks.

Example 3: Find the general solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y(x) = xe^{-x}.$$

Solution: We first find the complementary solution $y_c(x)$ by solving

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y(x) = 0.$$

This gives

$$y_c(x) = C_1 e^{-x} + C_2 e^{2x}.$$

We now shift to the particular solution. At first glance, we seem to fine—the term xe^{-x} does not appear as a fundamental solution of the complementary problem. We have to be careful, however. The naive trial function form is

$$y_p^{(1)}(x) = Axe^{-x} + Be^{-x}$$

which *does* have terms which overlap with those in the complementary solution—in particular, the term e^{-x}. So, even though xe^{-x} is not a problem term, the trial function which comes from it is incomplete. As before we will not be able to find a new solution with it. We must multiply $y_p^{(1)}(x)$ by x to get

$$y_p(x) = Ax^2e^{-x} + Bxe^{-x}.$$

Since neither of the terms here are contained in the complementary solution, we will be able to solve for the constants. We compute

$$y_p'(x) = -Ax^2e^{-x} + (2A - B)xe^{-x} + Be^{-x}$$

and

$$y_p''(x) = Ax^2e^{-x} + (-4A + B)xe^{-x} + (2A - 2B)e^{-x}.$$

Plugging these into the differential equation gives

$$y_p''(x) - y_p'(x) - 2y_p(x) = -6Axe^{-x} + (2A - 5B)e^{-x} = xe^{-x}.$$

It follows that we have the system $-6A = 1$ and $2A - 3B = 0$ so that $A = -1/6$ and $B = -1/9$. The particular solution is therefore

$$y_p(x) = -\frac{1}{6}x^2e^{-x} - \frac{1}{9}xe^{-x}.$$

while the general solution is

$$y(x) = C_1e^{-x} + C_2e^{-2x} - \frac{1}{6}x^2e^{-x} - \frac{1}{9}xe^{-x}.$$

4 Variation of Parameters

Now consider being asked to solve the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = \sec(2x). \tag{7}$$

While this is the same form as the examples we have dealt with before, there is one important difference: We do not know a simple function which always yields $\sec(2x)$ upon repeated differentiation. So which form of trial function $y_p(x)$ should we use? (Guess!)

It should not take long to realize that this approach is not going to get us far. It will turn out the particular solution is not a simple function of our basis trigonometric functions; nevertheless, we would like to be able to figure out what $y_p(x)$ is!

The method we are going to employ is called *variation of parameters.* The basic idea of this method is that we will try to *construct* the particular solution by taking the complementary solution

$$y_c(x) = C_1 \sin(2x) + C_2 \cos(2x)$$

and allowing some variance in the parameters C_1 and C_2. This is similar to the technique used to prove, for instance, that a repeated root r for the homogeneous constant coefficient case also had a solution xe^{rx}. We assumed then that there was another solution of the form $y_2(x) = u(x)y_1(x) = u(x)e^{rx}$. In this case, we will need *two* functions, one corresponding to C_1 and one corresponding to C_2. We have

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1(x)\sin(2x) + u_2(x)\cos(2x).$$

The idea is now the same! We want to plug this into the differential equation (7) and see if we can figure out what $u_1(x)$ and $u_2(x)$ need to be.

Before we begin, however, we will need to make one crucial assumption. We have two functions to solve for—$u_1(x)$ and $u_2(x)$—but only one piece of information, the differential equation itself. This is not enough—we need another piece of information. But what other piece of information do we have?

In fact, we do not have much more to work with, so we will just *assume* the second piece of information. We have yet to assume anything about the relationship between $u_1(x)$ and $u_2(x)$. For reasons which will become clear in a few minutes lines, it will be most convenient to assume that

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0.$$

For our particular system, this corresponds to

$$u_1'(x)\sin(2x) + u_2'(x)\cos(2x) = 0. \tag{8}$$

We now plug our hypothesized form of $y(x)$ into the differential equation. We need to compute $y''(x)$, so we first compute the first derivative. We have

$$\begin{aligned}\frac{dy}{dx} &= u_1'(x)\sin(2x) + 2u_1(x)\cos(2x) + u_2'(x)\cos(2x) - 2u_2(x)\sin(2x)\\ &= 2u_1(x)\cos(2x) - 2u_2(x)\sin(2x)\end{aligned}$$

where we have used (8) to simplify the equation. We now have

$$\frac{d^2y}{dx^2} = 2u_1'(x)\cos(2x) - 4u_1(x)\sin(2x) - 2u_2'(x)\sin(2x) - 4u_2(x)\cos(2x).$$

We can now easily compute that

$$\frac{d^2y}{dx^2} + 4y(x) = 2u_1'(x)\cos(2x) - 2u_2'(x)\sin(2x) = \sec(2x).$$

Bringing everything together, we can see that we need to satisfy the system of equations

$$\begin{aligned} 2u_1'(x)\cos(2x) - 2u_2'(x)\sin(2x) &= \sec(2x) \\ u_1'(x)\sin(2x) + u_2'(x)\cos(2x) &= 0. \end{aligned} \tag{9}$$

We want to solve this system of two equations in two unknowns for $u_1'(x)$ and $u_2'(x)$. The most efficient method is through matrix algebra (although solving the two equations successively will also work). If you do not know any matrix algebra, you may skip to the formulas and verify that they work. We have that (9) can be represented in matrix form and as

$$\begin{bmatrix} 2\cos(2x) & -2\sin(2x) \\ \sin(2x) & \cos(2x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} \sec(2x) \\ 0 \end{bmatrix}.$$

$$\Longrightarrow \quad \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \frac{1}{2\cos^2(2x) + 2\sin^2(2x)} \begin{bmatrix} \cos(2x) & 2\sin(2x) \\ -\sin(2x) & 2\cos(2x) \end{bmatrix} \begin{bmatrix} \sec(2x) \\ 0 \end{bmatrix}.$$

It follows that we have the system

$$\begin{aligned} u_1'(x) &= \frac{1}{2}\cos(2x)\sec(2x) = \frac{1}{2} \\ u_2'(x) &= -\frac{1}{2}\sin(2x)\sec(2x) = -\frac{1}{2}\tan(2x). \end{aligned}$$

All right, now we are finally getting somewhere! We have expressions for the derivatives of the unknown functions $u_1(x)$ and $u_2(x)$ in terms of things we know. All that remains to do is integrate. We have

$$u_1(x) = \int \frac{1}{2}\, dx = \frac{x}{2}$$

and

$$u_2(x) = -\frac{1}{2}\int \tan(2x)\, dx = \frac{1}{4}\ln|\cos(2x)|.$$

Recalling that our form for $y_p(x)$ was $y_p(x) = u_1(x)\sin(2x) + u_2(x)\cos(2x)$, we have that

$$y_p(x) = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\ln|\cos(2x)|.$$

That's it! It can be verified (although, with a lot of work!) that this satisfies (7) so that we have found the correct form of the particular solution. It should be obvious at this point that, no matter how long we spent guessing trial function forms for $y_p(x)$, we would have been very unlikely to have ever stumbled upon this particular form. Variation of parameters was absolutely necessary for solving this problem. The general form of the solution is

$$y(x) = C_1 \sin(2x) + C_2 \cos(2x) + \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) \ln |\cos(2x)|.$$

It should also be obvious that this is not a procedure that we would like to go through every time we want to compute a particular solution. Fortunately, the procedure we have outlined here can be perform for the general solution for $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. The following result, which is proved in Section 3.6 of the text, allows us to skip most of the computation (but will not allow us to avoid the integration!).

Theorem 4.1. *Consider a nonhomogeneous linear differential equation of the form*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = g(x)$$

with fundamental solutions $y_1(x)$ and $y_2(x)$. Then the particular solution is given by

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}\, dx$$
$$u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}\, dx$$

where $W(y_1, y_2)(x)$ is the Wronkian of y_1 and y_2 evaluated at the point x. Furthermore, the general solution is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).$$

We should stop to make a few notes about these formulas:

- While this looks great in principle—we have an explicit formula for the particular solution!—in general, it can be *very* hard to solve the integrals required of $u_1(x)$ and $u_2(x)$ explicitly. If the method of undetermined coefficients is possible, it will save a significant amount of time to use it.

- This is one of the few times in this course where I will concede that it is actually easier to use the formulas than to remember the method. Having seen the method performed in detail once, we will now be content to use the general formulas.

- This method works even for linear differential equations which do not have constant coefficients. There is one subtle point to make here—we still need to have the fundamental solutions y_1 and y_2 to the homogeneous problem given to us, and they may not be known if the coefficients are not constants.

Example 1: Verify that the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = 12x$$

is $y_p(x) = 3x$ by using variation of parameters.

Solution: We now just need to identify the required factors in the original statement of the problem, and then apply the formulas. We know that $y_1(x) = \sin(2x)$, $y_2(x) = \cos(2x)$, and $g(x) = 12x$. We can easily compute that

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = -2\sin^2(x) - 2\cos^2(x) = -2.$$

It follows by the formulas that we have

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}\, dx \\ &= \frac{1}{2}\int 12x\cos(2x)\, dx \\ &= \frac{3}{2}(2x\sin(2x) + \cos(2x)) \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}\, dx \\ &= -\frac{1}{2}\int 12x\sin(2x)\, dx \\ &= -\frac{3}{2}(\sin(2x) - 2x\cos(2x)) \end{aligned}$$

where the integrals have been computed using integration by parts (not shown). The particular solution is then given by

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= \frac{3}{2}(2x\sin(2x) + \cos(2x))\sin(2x) - \frac{3}{2}(\sin(2x) - 2x\cos(2x))\cos(2x) \\ &= 3x(\sin^2(2x) + \cos^2(2x)) = 3x. \end{aligned}$$

So we have recovered our previous answer. It should be obvious at this point, however, that the method of undetermined coefficients, while less powerful, is *significantly easier to use* when it can be applied. In this case, we skipped most of the details, and still have to integrate by parts *twice* while remembering a pair of complicated formulas. Nevertheless, we should be happy that the two methods gave the answers. It would have been a shame to go through all that work to find out we have missed a sign, or forget which factor was placed into which formula.

Example 2: Use the method of undetermined coefficients to find the general solution of

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2$$

given that $y_1(x) = e^x$ and $y_2(x) = x$ are solutions of the corresponding homogeneous equation

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Solution: The difference with this problem is we do not have a simple method by which determine the solutions of the homogeneous equation, since the coefficients are non-constant. Nevertheless, we can check that $y_1(x) = e^x$ and $y_2(x) = x$ are solutions through direct substitution.

Given these two functions, however, the steps for obtaining the particular solution is the same. We first need to write the equation in standard form. We have

$$\frac{d^2y}{dx^2} + \left(\frac{x}{1-x}\right)\frac{dy}{dx} - \left(\frac{1}{1-x}\right)y = 1-x.$$

It follows that we have $y_1(x) = e^x$, $y_2(x) = x$, and $g(x) = 1 - x$. It follows that

$$\begin{aligned} W(y_1, y_2)(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^x(1) - e^x(x) \\ &= e^x(1-x). \end{aligned}$$

Consequently

$$\begin{aligned}
u_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}\, dx \\
&= -\int \frac{x(1-x)}{e^x(1-x)}\, dx \\
&= -\int xe^{-x}\, dx \\
&= (1+x)e^{-x}
\end{aligned}$$

where we have applied integration by parts on the final step. We also have

$$\begin{aligned}
u_2(x) &= \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}\, dx \\
&= \int \frac{e^x(1-x)}{e^x(1-x)}\, dx \\
&= \int (1)\, dx \\
&= x.
\end{aligned}$$

The particular solution is therefore

$$\begin{aligned}
y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\
&= 1 + x + x^2.
\end{aligned}$$

It follows that the general solution is

$$y(x) = C_1e^x + C_2x + 1 + x + x^2 = \tilde{C}_1e^x + \tilde{C}_2x + 1 + x^2.$$

Note in particular that it was important to write the equation is *standard form* before attempting to solve. Otherwise, we would have ended up with an incorrect answer (and probably not even been able to get past the integration step!).