MATH 319, WEEK 8:
Mechanical Systems, Resonance

1 Pendulum/Spring Model

Let's reconsider the pendulum/spring model from last week. We used Newton's second law $F = ma$ to derive the equation

$$m\frac{d^2x}{dt^2} = F_{restoring} + F_{friction} = -kx(t) - c\frac{dx}{dt} \tag{1}$$

which gives the second-order homogeneous linear differential equation with constant coefficients

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = 0. \tag{2}$$

We can now solve this equation! We can also interpret the solution of this equation. First of all, we have the guess solution $y(x) = e^{rt}$ yields

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

The only thing that is different than the general case is that the constants are assumed, for physical reasons, to be strictly positive (i.e. $m > 0$, $c > 0$, $k > 0$).

We have the following three cases:

1. **Overdamped**: If $c^2 > 4mk$ then (1) has the general solution

   $$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

2. **Critically damped**: If $c^2 = 4mk$ then (1) has the general solution

   $$x(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

3. **Underdamped**: If $c^2 < 4mk$ then (1) has the general solution

   $$x(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

   where $\alpha = Re(r)$ and $\beta = Im(r)$.

Note: The positivity of the physical constants guarantees that either $r < 0$ or $\alpha < 0$ so long as $c > 0$. (Notice that this is not necessarily true for the general case.) This guarantees that the exponential is always a *decreasing* exponential. This says that the solution is always decaying toward its resting state, as we would expect from a *damped* pendulum or spring.

Note: It is easy to see where the classifications (overdamped, critically damped, and underdamped) come from. In Case (1) the damping parameter exceeds the other combined parameters ($c^2 > 4mk$), in Case (2) they are equal ($c^2 = 4mk$), and in Case (3) the other parameters exceed the damping ($c^2 < 4mk$).

Note on physical units: In order to associate (2) with actual physical models, we will need to give units to the variables and parameters. To make computations as straight-forward as possible, we will consider the units meters (m), kilograms (kg), and seconds (s) for length, mass, and time, respectively. The question, remains, however, of what the units of the *parameters* c and k are. To answer this (as best we can), we recall that a *Newton* is defined as

$$N = 1\ kg\frac{m}{s^2}.$$

This is the basic unit of *force*. We recall that (2) was derived from a force equation—consequently, the unit of each individual term in (2) (i.e. $mx''(t)$, $cx'(t)$ and $kx(t)$) must be Newtons! Since we know the units of $x(t)$ (m), $x'(t)$ (m/s) and $x''(t)$ (m/s^2) we see that the required units for c and k are

$$\begin{aligned} c &\sim kg/s = \frac{kg(m/s^2)}{m/s} = \frac{N}{m/s} \\ k &\sim kg/s^2 = \frac{kg(m/s^2)}{m} = \frac{N}{m}. \end{aligned}$$

Thus we will give the restoring constant k in terms of *Newtons per meter* and the frictional constant c in terms of *Newtons per unit velocity* or *Newtons per meter per second.*

Note on periodic solutions: In solutions to simple mechanical systems, we often encounter the form $C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$, which represents some sort of periodic motion. What is not obvious from this form, however, is that this is actually equivalent to a *single* phase-shifted trigonometric function with a different amplitude. For instance, we can easily check that

$3\cos(t) + 4\sin(t)$ is the same as $5\cos(t - 0.927)$ (graph it!). In general, we always have that

$$C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A\cos(\omega_0 t - \alpha)$$

for some A and α. We will see how to compute A and α through examples. (You may have already been introduced to the method in a calculus course.)

Although the exponential dominates the long-term behavior (check by taking the limit as $t \to \infty$!), there are important qualitative differences between the three cases. The difference comes in *how* the solutions approach the resting state.

1. In Case 1, after a short transient period, solutions approach $x = 0$ *monotonically*. That is to say, solutions settle into a trajectory which consistently gets closer as time passes—each second they are closer to $x = 0$ than the last. (Note that trajectories may initial overshoot the resting position if the initial velocity is sufficiently high.)

2. In Case 2, solutions again settled into a trajectory which consistently gets closer to the resting position as time passes, but it takes longer to settle into that trajectory. In fact, it takes the maximal amount of time—if it takes any longer, it will enter into Case 3.

3. In Case 3, solutions *oscillate* as they approach $x = 0$. On average, the solutions approach the resting position, but they continually overshoot the resting position and then bounce back, and overshoot again. Notice that these oscillations continue forever!

The three cases are illustrated by Figure 1. Notice how the exponential dominates in all three cases. Even in Case 3, where solutions oscillate continuously, we may obtain important information about the way in which solutions approach $x = 0$ by bounding the exponential portion of the solution.

Example 1: Consider a 2 kg weight attached to the end of a spring which requires a force of 8 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 2 meters to the right and released with an initial velocity of 2 meters per second to the right, find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form

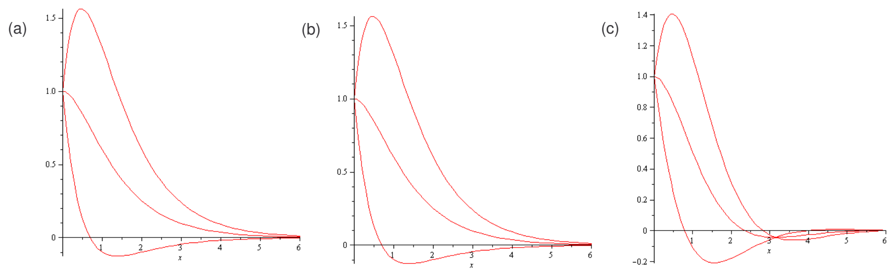

Figure 1: Example trajectories of the displacement of a damped pendulum/spring for the three cases: (a) overdamped; (b) critically damped; and (c) and underdamped. Notice that oscillations occur in the underdamped case.

$x(t) = A\cos(\omega_0 t + \alpha)$.

Solution: The given information implies that $m = 2$, $k = 8$ and $c = 0$. This gives the model

$$2\frac{d^2x}{dt^2} + 8x(t) = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 8) = 2e^{rx}(r^2 + 4) = 0$$

so that $r = \pm 2i$. It follows that the general solution has the form

$$x(t) = C_1\cos(2t) + C_2\sin(2t).$$

To find the particular solution satisfying the initial conditions, we must compute

$$x'(t) = -2C_1\sin(2t) + 2C_2\cos(2t).$$

The initial conditions give

$$\begin{aligned} x(0) = 2 \quad &\Longrightarrow \quad C_1 = 2 \\ x'(0) = 2 \quad &\Longrightarrow \quad 2C_2 = 2 \quad \Longrightarrow \quad C_2 = 1. \end{aligned}$$

It follows that the particular solution is

$$x(t) = 2\cos(2t) + \sin(2t).$$

We want to put the solution in the form $x(t) = A\cos(\omega_0 t - \alpha)$. What we need to do is expand $A\cos(\omega_0 t - \alpha)$ according to

$$A\cos(\omega_0 t - \alpha) = A\cos(\alpha)\cos(\omega_0 t) + A\sin(\alpha)\sin(\omega_0 t).$$

Comparing with the original equations gives $\omega_0 = 2$, and the system

$$\begin{aligned} C_1 &= A\cos(\alpha) \\ C_2 &= A\sin(\alpha). \end{aligned} \tag{3}$$

If we square these equations, add them, and then simplify we get

$$A = \sqrt{C_1^2 + C_2^2}.$$

Furthermore, we can divide the equations to get

$$\alpha = \tan^{-1}\left(\frac{C_2}{C_1}\right).$$

(Note that we may have to adjust α by a factor of π depending on which quadrant it is in. It is a good idea to take the answer here and check with the system (3).)

For our example, we have $C_1 = 2$ and $C_2 = 1$ so that $A = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\alpha = \tan^{-1}(1/2) \approx 0.4636$. We can check that this satisfies $\sqrt{5}\cos(0.4636) = 2$ and $\sqrt{5}\sin(0.4636) = 1$ so that we do not need to adjust by a factor of π. It follows that the solution can be written

$$x(t) = \sqrt{5}\cos(2t - 0.4636).$$

See Figure 2(a) for a plot.

Example 2: Reconsider the set-up provided in Example 1, but assume there is a damping of 4 Newtons for each meter/second of velocity. Find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form $x(t) = A(t)\cos(\omega_0 t + \alpha)$.

Solution: The given information tells us that we have $m = 2$, $k = 8$, and $c = 4$. This gives the model

$$2\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x(t) = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 4r + 8) = 2e^{rx}(r^2 + 2r + 4) = 0$$

which implies

$$r_{1,2} = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i.$$

The general solution is given by

$$x(t) = e^{-t}(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t).$$

In order to determine the particular solution, we must find $x'(t)$. We have

$$\begin{aligned} x'(t) = & -e^{-t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right) \\ & + \sqrt{3}e^{-t}\left(-C_1 \sin(\sqrt{3}t) + C_2 \cos(\sqrt{3}t)\right). \end{aligned}$$

The initial conditions result in the system

$$\begin{aligned} C_1 \qquad\quad &= 2 \\ -C_1 + \sqrt{3}C_2 &= 2. \end{aligned}$$

It follows that $C_1 = 2$ and $C_2 = \frac{4}{\sqrt{3}}$. It follows that the particular solution is

$$x(t) = e^{-t}\left(2\cos(\sqrt{3}t) + \frac{4}{\sqrt{3}}\sin(\sqrt{3}t)\right).$$

A more insightful form of this equation is to write it as

$$x(t) = A(t)\cos\left(\omega_0 t + \alpha\right).$$

As before, we have $A = \sqrt{C_1^2 + C_2^2} = \sqrt{2^2 + (4/\sqrt{3})^2} \approx 3.0551$ and $\alpha = \tan^{-1}(C_2/C_1) = \tan^{-1}((4/\sqrt{3})/2) = \tan^{-1}(2/\sqrt{3}) = 0.8571$. We can easily check that $3.0551\cos(0.8571) = 2$ and $3.0551\sin(0.8571) = \frac{4}{\sqrt{3}}$ so that we do not need to adjust by a factor of π. It follows that the solution can be written as

$$x(t) = 3.0551e^{-t}\cos\left(\sqrt{3}t - 0.8571\right).$$

See Figure 2(b) for a plot. It is worth noticing two things there:

1. Although the damped solution still oscillates, because of the additional exponential term e^{-t} we are guaranteed that eventually the amplitude of the solution will decay to zero. This corresponds to the pendulum or spring settling down to its equilibrium position, as we would expect in any realistic model.

2. The *quasi-frequency* of the damped oscillations is $\mu = \sqrt{3}$ which is *smaller* than the corresponding undamped value $\omega_0 = 2$. This means that the wavelength is *longer*. Specifically, we have that the period of the damped motion is $\sqrt{3}T = 2\pi$ so that $T = (2/\sqrt{3})\pi$. This is greater than the wavelength of the undamped motion, which was $T = \pi$. It follows that the addition of damping to the mechanism has *slowed* the oscillations down. This might have been expected—friction impedes motion, after all—but it will have important consequences in a moment.

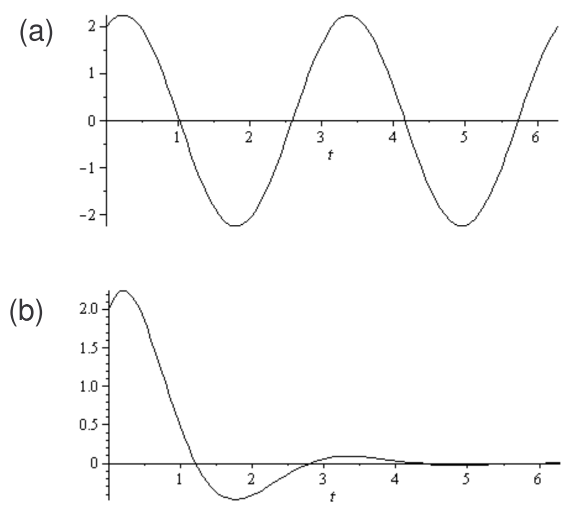

Figure 2: Behavior of an undamped (a) and damped (b) pendulum with mass 2 kg and restoring for 8 N/m. Notice that the initial conditions $x(0) = 2$ and $x'(0) = 2$ are the same for both simulations and coincides with the behavior of the plotted trajectories at $t = 0$.

Example 3: Now let's consider what happens to this system with the damping coefficient left general. That is to say, let's consider the behavior of the general system

$$2\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 8x(t) = 0.$$

Performing our standard analysis, we arrive at the quadratic

$$2r^2 + cr + 8 = 0$$

which gives

$$r = \frac{-c \pm \sqrt{c^2 - 64}}{4}.$$

We make a few observations at this point.

1. We notice first of all that there is a transition in the solution for at $c^2 = 64$, i.e. $c = 8$. Below this value (i.e. $c < 8$) the solution oscillates while above this value (i.e. $c > 8$) the solution does not. The value $c = 8$ is called the critical damping value.

2. We notice first of all that, so long as $c > 0$ (that is to say, so long as there is damping) then the solution will have negative exponentials. (This is trivial for the complex and repeated real root case, since the real part of r is $-c/4 < 0$. For the two distinct real roots case, it follows from the fact that $\sqrt{c^2 - 64} < |c|$ for $c > 8$.) We can therefore say that, regardless of the damping value, the long-term behavior of the solution is always convergence toward zero. In other words, the pendulum/spring *always* settles down if there is damping.

3. A quick analysis of the range $0 \leq c < 8$ reveals that the lengthening of the quasi-frequency as the damping increases is a general property. We have that the quasi-frequency is given by

$$\mu = \text{Im}(r) = \frac{\sqrt{64 - c^2}}{4}$$

which converges to zero as $c \to 8^-$. Notice that the wavelength is inversely related to the quasi-frequency, so that we can say the wavelength grows to infinity as $c \to 8^-$. This tells us exactly what is happening at the critical damping value $c = 8$! The wavelength lengthens as the damping increases, and if we reach the critical damping value the system's ability to oscillate at all will be eliminated—because the wavelength of oscillation has reached infinity.

2 Resonance

Now let's consider what happens when we add a *forcing term* to the (low-amplitude) pendulum and spring model. That is to say, in our force equation from before, we allow

$$F = F_{restoring} + F_{friction} + F_{forcing} = -kx(t) - c\frac{dx}{dt} + f(t)$$

where $f(t)$ represents a forcing term *external* to the system. This could represents shaking the pendulum/spring manually or having the pendulum/spring connected to some bigger machinery. This gives rise to the *non-homogeneous* second-order differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = f(t).$$

We will typically assume that the forcing is *sinuisoidal*—that is to say, we will assume the forcing $f(t)$ can be represented by some combination of sines and cosines. This represents shaking the undamped pendulum or spring with a fixed frequency.

Consider the example of solving the following differential equation (corresponding to the physical model) with initial conditions $x(0) = 0$ and $x'(0) = 0$:

$$\frac{d^2x}{dt^2} + 4x(t) = \cos(\omega t)$$

where $\omega \neq 2$. (This corresponds to the pendulum system for a mass of 1 kg and a restoring constant k of 4 Newtons per meter. The parameter ω controls the frequency of the shaking and the initial conditions correspond to starting the system at rest.)

We can have already seen that the complementary function for this differential equation is

$$x_c(t) = C_1\cos(2t) + C_2\sin(2t).$$

Since $\omega \neq 2$, we use the trial function $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$. This gives

$$x_p''(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

so that we have

$$\begin{aligned} x_p''(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) + 4(A\cos(\omega t) + B\sin(\omega t)) \\ &= (4-\omega^2)(A\cos(\omega t) + B\sin(\omega t)) \\ &= \cos(\omega t). \end{aligned}$$

Since $\omega \neq 2$ implies $\omega^2 \neq 4$, it follows that $A = 1/(4-\omega^2)$ and $B = 0$ so that we have the general solution

$$x(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4-\omega^2}\cos(\omega t).$$

This has derivative

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - \frac{\omega}{4-\omega^2} \sin(\omega t)$$

so that the initial conditions $x(0) = 0$ and $x'(0) = 0$ give the system

$$\begin{aligned} C_1 &= -\frac{1}{4-\omega^2} \\ 2C_2 &= 0 \end{aligned}$$

which implies $C_1 = -1/(4-\omega^2)$ and $C_2 = 0$. It follows that the solution is

$$\begin{aligned} x(t) &= -\frac{1}{4-\omega^2} \cos(2t) + \frac{1}{4-\omega^2} \cos(\omega t) \\ &= \frac{1}{4-\omega^2} \left(\cos(\omega t) - \cos(2t)\right). \end{aligned}$$

In terms of simplification, this is pretty good, but in fact we can do a little better. The trigonometric identities $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ can be subtracted from one another to give $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$. If we take $A = \frac{1}{2}(2+\omega)t$ and $B = \frac{1}{2}(2-\omega)t$ we have

$$A + B = 2t, \quad \text{and} \quad A - B = \omega t.$$

Remarkably (but usefully?), this implies that the solution can be written as the single term

$$x(t) = \frac{2}{4-\omega^2} \sin\left(\frac{1}{2}(2+\omega)t\right) \sin\left(\frac{1}{2}(2-\omega)t\right).$$

Okay, this is getting a little ridiculous. What is the point of all this algebra? Well, this is actually *incredibly* insightful for of the solution. We now have the solution decomposed into two sine functions with different frequencies (corresponding to the difference in the natural and forcing frequencies!). In particular, if ω is near 2, there is a separation of time-scales in the two modes. We have that

1. There is a *slow* oscillatory mode with wavelength $(4\pi)/(2-\omega)$. This mode can be thought of as an envelop which restricts all other modes (since all other modes must multiply through this function, so can only be as big as this slow mode allows it to be). (See Figure 3(a))

2. There is a *fast* oscillatory mode with wavelength $(4\pi)/(2+\omega)$. This mode oscillates faster than the other mode but is restricted through each period by its slower counterpart.

3. Since sine is bound by -1 and 1, the maximal amplitude of the solution is $2/(4-\omega^2)$.

This raises a very interesting question: What happens as the forcing frequency is *changed* related to the fixed natural frequency of the system (i.e. the frequency the undamped pendulum or spring swings when left alone)? In particular, what happens as $\omega \to 2$?

We can consider this as ω approaches 2 from either side, since the separation of time-scales holds. We make the following observations:

1. The ω approachs 2, the wavelength of the slow mode *explodes* while the wavelength of the fast mode stays roughly the same. That is to say, the separation in time-scales intensifies in that the number of times the fast mode completes its cycle before the slow mode completes its cycle becomes unbounded. (See Figure 3(b).)

2. The amplitude $2/(4-\omega^2)$ also *explodes*. In fact, in the limit, we have that the amplitude is infinite. (See Figure 3(c).)

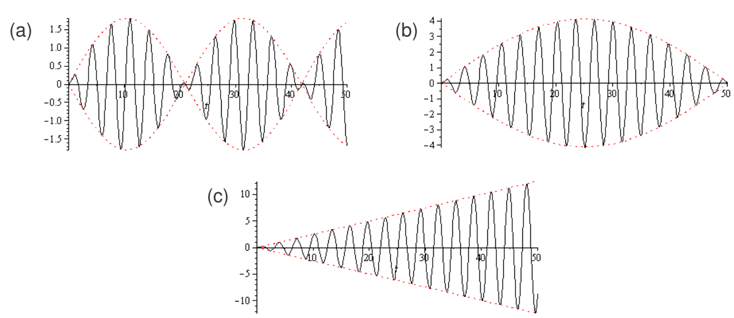

Figure 3: Solution of the mechanical system with sinusoidal forcing with (a) $\omega = 1.7$; (b) $\omega = 1.875$; and (c) $\omega = 1.99$. Notice that the y-axis grows as ω gets closer and closer to 2.

Something seems to be going incredibly wrong in this example. How can we have the amplitude of our solution explode to infinity? Worst still, we know that the solution still oscillates by a fixed period, so as time goes on and one the solution (i.e. the pendulum or spring) will make jumps from the positive extreme to the negative extreme *in the same amount of time*! What is going on?

Let's reconsider our physical example. What is really happening as ω approaches 2? Recall that 2 is the natural frequency term for the *underlying* system. Is the term controlling how the body naturally oscillates if simply let go. Now imagine shaking that in a very particular way—and in particular, in a way that is completely *in phase* with the natural rhythm of the body. Well, then, every time the pendulum naturally wants to kick left, we give it an extra push, and every time it wants to kick right, we give it an extra push in that direction, too. If we do this exactly in sync with the body's natural rhythm, we imagine that the amplitude will certainly grow!

Before we get carried away with this example too far, we should recognize that there are certain physical constraints (e.g. damping, whether in the form of friction or something else). We also neglected other physical concerns. For a pendulum, for example, we will swing over the top far before we will extend off to infinite in any diretion. And, for a spring, we imagine that if we compress or overextend the spring too much it will simply *break* rather than extend to infinite length. Nevertheless, this is an interesting phenomenon to investigate and is a concern in many applications. What we have discovered is **resonance**.

We might wonder what has happened with our solution. After all, we cannot very well have $x(t) = \infty$ as a meaningful solution. Rather, the solution breaks down, but if we consider the original differential equation we immediately see why. If we have $\omega = 2$ we are in the case where we may not use a trial function of the form $x_p(t) = A\cos(2t) + B\sin(2t)$. We have already solved this using the trial function $x_p(t) = At\cos(2t) + Bt\sin(2t)$ and got

$$x(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}t\sin(2t).$$

The initial conditions $x(0) = 0$ and $x'(0) = 0$ give $C_1 = C_2 = 0$ give the simple solution

$$x(t) = \frac{1}{4}t\sin(2t).$$

Just as we expected, we have a solution which oscillates with increasing amplitude (as t grows). In other words, we have filled in the gap in our previous physical reasoning. Even though the solution methods were completely different, the limit of the previous solution approaches this resonate solution as ω approaches 2!