MATH 319, WEEK 9:
Power Series Methods

1 Power Series Methods

Consider being asked to solve the differential equation

$$xy''(x) - (1 + x)y'(x) + y(x) = 0.$$

The key difference between this second-order differential equation and the previous ones we have considered is that the coefficients are not constants—rather, they vary with x. While this may not seemed like a significant change, it will prove disastruous for our previous intuition of guessing specific functional forms and cleaning things up as appropriate. *There is no general solution method for second-order differential equations with variable coefficients.* (It should be noted that this is *not* to say that solutions do not exist, or are not easy to write down, just that there is no general way to find them.)

We will have to rely on alternative analysis methods, but what? Numerical methods immediately come to mind, but we do not currently have the tools for such methodology for second-order systems (yet!) since these methods depended implicitly on having first-order derivatives. Consider instead the following intuition:

1. Suppose the solution $y(x)$ has a power series expansion

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

2. If it does, we can differentiate this form term-by-term to obtain series expansions for $y'(x)$, $y''(x)$, and so on.

3. We can then plug these power series forms into the differential equation, rearrange, and solve for the coefficients a_n.

4. Wherever the power series converges, we can obtain an estimate for the value of the solution at that point by computing a sufficient number of terms in the expansion.

On the face of it, this seems very similar (but more work!) than a numerical method because we cannot generally evaluate an infinite sum and so may only typically *estimate* the value of interest by taking a truncated sum. We will see that this method can also on occasion lead to a general analytic solution as well.

Let's see how this might play out for our example. For simplicity, we will assume the power series is centered at $x_0 = 0$. (This value of x_0 will generally be given to us.) So we are looking for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Differentiating term-by-term yields

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots \end{aligned}$$

where we have removed from the sum the terms which evaluate to zero. We can now plug this into the left-hand side of the differential equation to get

$$\begin{aligned} &xy''(x) - (1+x)y'(x) + y(x) \\ &= x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - (1+x) \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

We will have to be a little bit careful at the point with our indexing, and how we split our sums. We want to move the terms x and $1 + x$ inside the summations. We notice that they multiply *every* term in the associated sums, so that we can actually just float them into the summations directly. We have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

We now want to collect like terms according to their powers of x. This cannot be done directly, since two of the sums are with respect to x^{n-1} and two are with respect to x^n. To resolve this, we will need to reindex two of the summations to match the others. This can be a tricky task in general! In fact, I would go so far as to say it is not even obvious that we should be able to do this at all. But we can easily check that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = 2a_2 x + 6a_3 x^2 + 12a_4 x^3 + \cdots$$

and

$$\sum_{n=1}^{\infty} (n+1)na_{n+1} x^n = 2a_2 x + 6a_3 x^2 + 12a_4 x^3 + \cdots$$

give *exactly* the same series. It should also be now clear how they were correspond. If we want to shift the index inside the summation (i.e. the n) *up* we need to shift the starting point for the summation *down*. A similar rule applies for shifting interior indices *down* (the external bounds must be shifted *up*).

It should not take much convincing to see that the original series can in fact be written as

$$\sum_{n=1}^{\infty} (n+1)na_{n+1} x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

This is terrific! But we are not done yet. The summations do not begin at the same index, so we cannot yet combine them. To resolve this (small) problem all we need to do is *take out* all of the terms below the lowest common starting point for the sums. That is to say, because our sums start at either $n = 0$ and $n = 1$, we simply take out all of the terms in the sums corresponding to $n = 0$. (Note that only two of the sums have terms corresponding to $n = 0$!) This gives us

$$-a_1 + a_0 + \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n - \sum_{n=1}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=1}^{\infty} a_n x^n$$

We may now (finally!) combine the summations. We have

$$\begin{aligned} &a_0 - a_1 + \sum_{n=1}^{\infty} \left[(n+1)na_{n+1} - (n+1)a_{n+1} - na_n + a_n\right] x^n \\ &= a_0 - a_1 + \sum_{n=1}^{\infty} \left[(n+1)(n-1)a_{n+1} + (1-n)a_n\right] x^n = 0. \end{aligned} \tag{1}$$

Noticing that the $n = 1$ term evaluates to zero, what this expression is really telling us is that

$$(a_0 - a_1) + (3a_3 - a_2)x^2 + (8a_4 - 2a_3)x^3 + \cdots = 0.$$

The only way for this to hold is if *every coefficient is equal to zero.* That is to say, we need to have

$$a_0 - a_1 = 0, 3a_3 - a_2 = 0, 8a_4 - 2a_3 = 0, \text{etc.}$$

It would be a lot of work to do this for each term individual. We notice, however, that our general form (1) gives a far more general form corresponding to the coefficients being equation to zero. We have that

$$a_0 - a_1 = 0$$

and the far more important

$$(n+1)(n-1)a_{n+1} + (1-n)a_n = 0 \quad \Longrightarrow \quad a_{n+1} = \frac{a_n}{n+1}$$

where the last form is valid for $n \geq 2$ only. In other words, we can explicitly relate each coefficient in the power series expansion to the previous one by a *recurrence relation.*

There are several possibilities at this point, which will depend sensitively on the particular problem under consideration and the available resources at hand (e.g. is a computer handy?):

1. We may be interested only the first few terms of the power series for $y(x)$ (e.g. four or five terms). In that case, we will compute a_0 through a_4 or a_5 and neglect the rest of the terms.

2. We may be able to find a general solution for as a result of the recurrence relation, in which case we are looking for an explicit closed form for a_n. This form is usually in terms of a_0 or a_1.

3. We may be able to identify the relationship between the a_n, but not be able to give a closed form for it. In this case, we just do the best we can.

In this case, we will try to find the general form for a_n. That is to say, we want to solve the recurrence relation. We start by considering $a_0 - a_1 = 0$, which immediately gives us $a_1 = a_0$. We now consider the general recurrence relation. We have

$$a_{n+1} = \frac{a_n}{n+1} \quad \implies \quad a_n = \frac{a_{n-1}}{n}.$$

We can see that, increasing n, we have

$$\begin{aligned} n = 3 \quad &\implies \quad a_3 = \frac{a_2}{3} \\ n = 4 \quad &\implies \quad a_4 = \frac{a_3}{4} = \frac{a_2}{4 \cdot 3} \\ n = 5 \quad &\implies \quad a_5 = \frac{a_4}{5} = \frac{a_2}{5 \cdot 4 \cdot 3} \\ &\vdots \end{aligned}$$

We might conjecture at this point that the general form of the term is

$$a_n = 2 \cdot \frac{a_2}{n!}.$$

This can be proved by induction, but we will not perform this task, instead taking the fact as obvious. This gives the general form for the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0(1+x) + 2a_2 \sum_{n=2}^{\infty} \frac{x^n}{n!}.$$

This is the power series solution representation of $y(x)$.

We might notice, however, that the final sum is very close to the Taylor series expansion for e^x. In fact, it only differs in the first two terms $n = 0$ and $n = 1$, which are missing from the sum. We can be somewhat creative, therefore, and *complete* the sum. To do this, we need to add $1 + x$ to the sum, which means we need to subtract it at the same time. This gives

$$\begin{aligned} y(x) &= a_0(1+x) + 2a_2 \left[(-1-x) + 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \right] \\ &= a_0(1+x) + 2a_2 \left[(-1-x) + \sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \\ &= (a_0 - 2a_2)(1+x) + 2a_2 \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

This may not seem like much, but it is actually pretty remarkable! The terms a_0 and a_2 are undetermined constants, and the summation corresponds to the Taylor series expansion of e^x. It follows that we have

$$y(x) = C_1(1+x) + C_2 e^x$$

where $C_1 = a_0 - 2a_2$ and $C_2 = 2a_2$. (This answer can be easily checked.) That is to say, we have successfully used the power series method to find the *general solution* of the differential equation. This should be remarkable, since we had no direct method for solving this particular differential equation. (Although we should be slightly discouraged by the amount of work it took!)

We should stop here to make a few notes:

- It is not generally the case that we will be able to correspond our final power series solution to analytic solutions of the form e^x, $\ln(x)$, etc. In general, having the answer in series form may be the best we can do.

- Notice that, as a result of the recurrence relationship, we will have our series written in terms of some (two, in our case) baseline constants which will not be solved for numerical. As in the example above, these will correspond to our undetermined constants in the general solution, and can be solved for numerically by appropriate initial conditions.

- It may be very difficult to determine an explicit form for the general terms a_n. In such cases, it is more common to seek out the first three or four terms, and disregard the rest.

2 General Properties of Power Series

We should pause to review some general properties about power series. The first thing we should probably recall is that all of the elementary functions have expansions in terms of a special kind of power series known as a *Taylor series*, i.e. a series of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

That is to say, the power series with $a_n = f^{(n)}(x_0)/n!$. We have the following well-known Taylor series expansions:

$$\begin{aligned}
e^x &= \sum_{n=0}^{n} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \\
\ln(1-x) &= -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots, \qquad -1 \le x < 1 \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1.
\end{aligned}$$

We should also remind ourselves that power series are not guaranteed to converge for all $x \in \mathbb{R}$. They may have a limited *radius of convergence*, $|x - x_0| < \rho$, outside of which the series does not settle down as $n \to \infty$ (i.e. as we take more terms). The most common test for the convergence of power series is the ratio test, which says, if we evaluate

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| L$$

then the series converges if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. The endpoints, where $|x - x_0|L = 1$, have to be considered separately since the test does not apply to them.

Example: Show that the Taylor series expansion given above for e^x converges for all x while the expansion for $\ln(1-x)$ converges only for $-1 \le x < 1$.

Solution: The series for e^x has $a_n = 1/n!$ so we compute

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| &= |x| \lim_{n\to\infty} \left| \frac{1/(n+1)!}{1/n!} \right| \\
&= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.
\end{aligned}$$

Since this is clearly less than 1 for any value of x we could happen to choose, we have that the series converges for all $x \in \mathbb{R}$. For the series $\ln(1-x)$, we have $a_n = -1/n$ so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x| \lim_{n\to\infty}\left|\frac{(1/(n+1))}{1/n}\right| = |x| \lim_{n\to\infty}\frac{n}{n+1} = |x|.$$

Clearly, we have that this is only less than 1 if $|x| < 1$ so that the interval of convergence is 1. To check the endpoints, when $|x| = 1$, we have to the consider the series exactly. For $x = -1$ we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

which converges. For $x = 1$ we have

$$-\sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges (harmonic series). It follows that the interval of convergence is $-1 \leq x < 1$ and we are done.

For this course, we will not need to consider many more details of power series, except to recall that in order to add, subtract, or multiply two or more power series, or to differentiate or integrate a single power series, all we need to do is apply the operations *term-by-term*, and the resulting series will converge on the same interval (except perhaps the end points). That is to say, power series are very easy to manipulate!

2.1 Ordinary Points and Initial Value Problems

We will not investigate the theory underlying power series methods in depth, but it will be important to make one clarification about the general second-order differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{2}$$

It is reasonable to ask what conditions are sufficient for the method to work. After all, we don't want to be wasting our valuable time performing operations we should have realized were doomed to failure at the start!

Fortunately, the answer is very simple.

Lemma 2.1. *Suppose $P(x)$, $R(x)$, and $Q(x)$ are polynomials with no common factors and $P(x_0) \neq 0$. Then there is a neighbourhood of x_0, $|x - x_0| < \rho$, in which a power series solution of (2) exists and converges.*

The justification depends on the existence and uniqueness theorem for second-order linear differential equations, which we did not cover in class, but which can be found in Section 3.2 of the text (Theorem 3.2.1). We made the following notes:

- The points $x \in \mathbb{R}$ such that $P(x) \neq 0$ are called **ordinary points**. The points $x \in \mathbb{R}$ such that $P(x) = 0$ are called **singular points**.
- The important connection to make with Theorem 3.2.1 is with regards to *initial value problems*. This result tells us that, if x_0 is an ordinary point, then the initial value problem for the power series centered at x_0 can be solved. It will be important, therefore, to center our power series at ordinary points if we can!

Example 1: Show that the initial value problem

$$xy''(x) - (1 + x)y'(x) + y(x) = 0, \quad y(0) = a, y'(0) = b$$

is ill-posed (i.e. cannot be solved). Explain this in terms of Lemma 2.1.

Solution: We have that the solution can be represented as a power series

$$y(x) = a_0(1 + x) + 2a_2 \sum_{n=2}^{\infty} \frac{x^n}{n!}.$$

We suspect that, since there are two unsolved constants, a_0 and a_2, that we will be able to solve for them uniquely by applying initial conditions. We notice that we have

$$\begin{aligned} y'(x) &= a_0 + 2a_2 \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= a_0 + a_2 \sum_{n=1}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

We can see that $y(0) = a$ implies $a_0 = a$ while $y'(0) = b$ implies $a_0 = b$. This is an inconsistent system, since a_0 only has a consistent value if $a = b$, and a_2 is never determined.

But what has gone wrong? Recall our lemma! We were only *guaranteed* to have a solution to the initial value problem around a point x_0 if that point was an *ordinary* point. In this case we centered around $x_0 = 0$ and we have that $P(x_0) = x_0 = 0$ so that x_0 is a singular point. We should not be surprised to find that our initial value problem is ill-posed at this point. (In fact, we should be surprised that we were able to even obtain a solution using that point!)

Example 2: Find the first six non-zero terms in the power series solution (centered at $x_0 = 1$) of the following initial value problem

$$y''(x) - xy(x) - y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

Use this to estimate the value of $y(2)$.

Solution: We first of all note that $P(x) = 1$ so that $x_0 = 1$ (and in fact, any point) is an ordinary point so that we will be able to find a solution centering there. We assume the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - 1)^n$$

to get

$$y'(x) = \sum_{n=1}^{\infty} a_n n (x - 1)^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)(x - 1)^{n-2}.$$

Substituting this into the left-hand side of the differential equation gives

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - x \sum_{n=1}^{\infty} a_n n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n.$$

It is tempting to immediately carry the stray x term into the corresponding summation; however, this would produce terms of the form $x(x-1)^{n-1}$ and *not* the desired form $(x-1)^n$. In order to resolve this, we must rewrite x in factors of the power term $x - 1$. In this case, we can simply write

$x = (x-1)+1$ to get

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - [(x-1)+1]\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n$$
$$= \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} a_n n(x-1)^n$$
$$- \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n.$$

We now reindex the sums so that they have common terms $(x-1)^n$. This gives

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=1}^{\infty} a_n n(x-1)^n$$
$$- \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n.$$

To combine the terms, we may start no lower than $n = 1$, so we much extra the terms corresponding to $n = 0$ from the sums. After a little rearranging, this gives

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - (n+1)(a_{n+1}+a_n)\right](x-1)^n = 0.$$

Equating coefficients on the left-hand and right-hand sides gives

$$2a_2 - a_1 - a_0 = 0 \quad \text{and} \quad a_{n+2}(n+2)(n+1) - (n+1)(a_{n+1}+a_n) = 0.$$

This simplifies to the general recursion relationship

$$a_n = \frac{a_{n-1}+a_{n-2}}{n}$$

for $n \geq 2$.

We are probably not going to be able to immediately identify a general solution for the terms a_n, but we can still (and will always be able to!)

determine the first few terms in the series. We have

$$\begin{aligned}
a_2 &= \frac{a_1 + a_0}{2} = \frac{a_1}{2} + \frac{a_0}{2} \\
a_3 &= \frac{a_2 + a_1}{3} = \frac{(\frac{a_1}{2} + \frac{a_0}{2}) + a_1}{3} \\
&= \frac{a_1}{2} + \frac{a_0}{6} \\
a_4 &= \frac{a_3 + a_2}{4} = \frac{(\frac{a_1}{2} + \frac{a_0}{6}) + (\frac{a_1}{2} + \frac{a_0}{2})}{4} \\
&= \frac{a_1}{4} + \frac{a_0}{6} \\
a_5 &= \frac{a_4 + a_3}{5} = \frac{(\frac{a_1}{4} + \frac{a_0}{6}) + (\frac{a_1}{2} + \frac{a_0}{6})}{5} \\
&= \frac{3}{20}a_1 + \frac{a_0}{15}.
\end{aligned}$$

In other words, we can determine (with a little work!) how each coefficient of the power series solutions depends on a_0 and/or a_1. In the end, we have the series

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= a_0 + a_1(x-1) + \left(\frac{a_1}{2} + \frac{a_0}{2}\right)(x-1)^2 + \left(\frac{a_1}{2} + \frac{a_0}{6}\right)(x-1)^3 \\
&\quad + \left(\frac{a_1}{4} + \frac{a_0}{6}\right)(x-1)^4 + \left(\frac{3}{20}a_1 + \frac{a_0}{15}\right)(x-1)^5 + \cdots \\
&= a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots\right] \\
&\quad + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots\right]
\end{aligned}$$

Factoring the solution by a_0 and a_1 allows us to make the immediate correspondence between the power series form and the general form

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

That is to say, it allows us to extract a fundamental solution set in the form of independent series!

This form also allows us to quickly evaluate the initial conditions: $y(1) = 0$ and $y'(1) = 1$. In general, this could be a substantial task. The solution is composed of infinite sums, but we suspect that we may have to evaluate an infinite sum when computing a_0 or a_1. We notice, however, that evaluating $x = 1$ immediately eliminates all of the factored forms $(x-1)^n$ from the summation! Once we remove these terms, we are left with

$$y(1) = 0 \implies a_0 = 0$$

and

$$y'(1) = 1 \implies a_1 = 1.$$

It follows that the particular solution is

$$y(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots$$

We can now easily estimate the value of $y(2)$ by evaluating

$$y(2) \approx (1) + \frac{1}{2}(1)^2 + \frac{1}{2}(1)^3 + \frac{1}{4}(1)^4 + \frac{3}{20}(5)^5 = \frac{12}{5} = 2.4.$$

This is close to the true value of $y(2) = 2.517182610$ but we should not be surprised that we will have to take significant more terms in order to get a truly "good" estimate. If be go up to the x^{10} term, we obtain the estimate $y(2) = 2.516316138$ which is accurate to two decimal places. Going up to the x^{20} term gives the estimate $y(2) = 2.517182608$, which is accurate to seven decimal places. And so on. We can see that the trade off being this and the earlier numerical methods is roughly the same: *the greater the accuracy desired, the greater the computation resources required.*