

MATH 319, WEEK 10: Laplace Transforms

1 Laplace Transforms

We now introduce a particularly important transformation in the theory of ordinary differential equations. The *Laplace transform* of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x) dx.$$

We will unfortunately not be able to motivate the utility of this transformation without a little more background work, but we will start by making the following notes:

- The general idea is that we will be able to use Laplace transforms to rewrite many differential equations in an alternative (but equivalent) form which is *easier to solve*. There are several advantages to the method, but the main ones are the following:
 1. The transformed problem may be solved by *algebraic methods alone*. There are no derivatives or integrals at all! (Partial fractions, in particular, with factor prominently.)
 2. The Laplace transform method is a *one-step* solution method. It incorporates initial conditions and nonhomogeneities directly into the procedure!
 3. Laplace transform methods are particularly well-adapted to handle problems with *discontinuous forcing functions*. (Recall that previously, if we had a discontinuous forcing function we had to solve problems independently in each interval, matching up boundary conditions as we went. This could be incredibly messy! We will see that Laplace transforms have a very natural way of handling discontinuous forcing functions.)
- Although the details will differ, the general idea will be the same as that of substitution methods. We will:
 1. Transform the differential equation (in y and x) into a new problem (in s).

2. Solve the new problem (an algebraic equation in the variable s !).
 3. Transform *back* to the original variables y and x .
- Notice that the Laplace transform is an *indefinite integral*. We will have to recall that

$$\int_0^{\infty} e^{-sx} f(x) dx = \lim_{A \rightarrow \infty} \int_0^A e^{-sx} f(x) dx$$

and the related theory. In particular, the integral may converge or diverge. The Laplace transform of a function only exists if the integral converges.

1.1 Laplace Transforms of Elementary Functions

We start by building a catalogue of Laplace transforms. Notice, importantly, that all of the Laplace transforms we will compute will amount to taking a function of x ($f(x)$) into a function of s ($F(s)$)!

Example 1: Compute the Laplace transform of $f(x) = c$ where c is some constant.

Solution: We just need to apply the definition. We have

$$\mathcal{L}\{c\} = \int_0^{\infty} ce^{-sx} dx = -c \lim_{A \rightarrow \infty} \left. \frac{e^{-sx}}{s} \right|_0^A = -c \left[\lim_{A \rightarrow \infty} \frac{e^{-sA}}{s} - \frac{1}{s} \right].$$

We can see that $\lim_{A \rightarrow \infty} e^{-sA}/s$ diverges to infinity if $s < 0$ (and the form was invalid for $s = 0$, which clearly diverges). Otherwise, it converges to zero. It follows that we have convergence for $s > 0$ only, so that

$$\mathcal{L}\{c\} = \frac{c}{s}, \quad s > 0.$$

That was pretty easy, but we would be understandably skeptical if we did not believe the Laplace transform would work out as clean as that for more complicated functions. Let's try another one.

Example 2: Compute the Laplace transform of $f(x) = e^{ax}$ where $a \neq 0$ is some constant.

Solution: Again, we just apply the definition. We have

$$\begin{aligned}\mathcal{L}\{e^{ax}\} &= \int_0^\infty e^{-sx} e^{ax} dx = \int_0^\infty e^{(a-s)x} \\ &= \lim_{A \rightarrow \infty} \left. \frac{e^{(a-s)x}}{a-s} \right|_0^A = \lim_{A \rightarrow \infty} \frac{e^{(a-s)A}}{a-s} - \frac{1}{a-s}.\end{aligned}$$

Again, the limit does not converge everywhere, but we can see that it does for $s > a$. It follows that we have

$$\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}, \quad s > a.$$

Example 3: Compute the Laplace transform of $f(x) = x$.

Solution: We have

$$\begin{aligned}\mathcal{L}\{x\} &= \int_0^\infty x e^{-sx} dx = \lim_{A \rightarrow \infty} \left[-x \frac{e^{-sx}}{s} \Big|_0^A + \frac{1}{s} \int_0^A e^{-sx} \right] \\ &= \lim_{A \rightarrow \infty} \left[-A \frac{e^{-sA}}{s} - \frac{e^{-sx}}{s^2} \Big|_0^A \right] \\ &= \lim_{A \rightarrow \infty} \left[-A \frac{e^{-sA}}{s} - \frac{e^{-sA}}{s^2} + \frac{1}{s^2} \right]\end{aligned}$$

by integration by parts. As we have seen before, these integrals converge to zero if $s > 0$ so that we have

$$\mathcal{L}\{x\} = \frac{1}{s^2}.$$

A common property of the derivation of Laplace transforms is that the resulting integrals will require integration by parts. The good news, is that this is capable of handling most of our standard functions! The details may be messy, but it can be checked that

$$\begin{aligned}\mathcal{L}\{x^n\} &= \frac{n!}{s^{n+1}}, \quad s > 0 \quad (n = 0, 1, 2, 3, \dots) \\ \mathcal{L}\{\sin(bx)\} &= \frac{b}{s^2 + b^2}, \quad s > 0 \\ \mathcal{L}\{\cos(bx)\} &= \frac{s}{s^2 + b^2}, \quad s > 0.\end{aligned}$$

It is also important to note that the Laplace transform can easily be applied to *linear combinations* of these functions. In general, we have

$$\begin{aligned}\mathcal{L}\{C_1f(x) + C_2g(x)\} &= \int_0^\infty e^{-sx}(C_1f(x) + C_2g(x)) dx \\ &= C_1 \int_0^\infty e^{-sx}f(x) dx + C_2 \int_0^\infty e^{-sx}g(x) dx \\ &= C_1\mathcal{L}\{f(x)\} + C_2\mathcal{L}\{g(x)\}.\end{aligned}$$

In this case, the region of validity in s is the smallest integral allow by each individual transformation. This will allow us to break big problems into smaller ones.

Example 4: Determine the Laplace transform of

$$f(x) = e^{-2x} + 3x^2 - \sin(5x).$$

Solution: We notice first of all that we may decompose this into smaller problems by the linearity property just discovered. We have

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{e^{-2x}\} + 3\mathcal{L}\{x^2\} - \mathcal{L}\{\sin(5x)\}.$$

We have rules for all of these forms! We can now easily compute that

$$\mathcal{L}\{f(x)\} = \frac{1}{s+2} + \frac{6}{s^3} - \frac{5}{s^2+25}.$$

Also notice that the first transformation is valid for $s > -2$ while the second two are valid for $s > 0$. It follows that the above transformation is valid for $s > 0$.

1.2 Inverse Laplace Transforms

A key portion of what we will doing is converting back from the Laplace tranform world to standard functions of y and x . This naturally means we will have to *invert* the Laplace transform, i.e. we will have to compute

$$\mathcal{L}^{-1}\{F(s)\} = f(x).$$

Formally defining the operation would be incredibly tedious, but it is also going to be unnecessary. We already know the basic forms of a number of Laplace transforms! Just like the relationship between derivation and

integration, we are simple going to make the following table and correspondences:

$$\begin{aligned} \mathcal{L}\{e^{ax}\} = \frac{1}{s-a} &\implies \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax} \\ \mathcal{L}\{x^n\} = \frac{n!}{s^{n+1}} &\implies \mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\} = x^{n-1} \\ \mathcal{L}\{\sin(bx)\} = \frac{b}{s^2+b^2} &\implies \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bx) \\ \mathcal{L}\{\cos(bx)\} = \frac{s}{s^2+b^2} &\implies \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos(bx) \end{aligned}$$

We will add to this list later, but it will be important to go through a few examples now to see what subtleties may arise.

Example 1: Determine the inverse Laplace transform of

$$F(s) = \frac{24}{s^4} - \frac{9}{s^2+9}.$$

Solution: It is important first of all to recognize the linearity of the Laplace transform also applies to the inverse Laplace transforms. That is to say, we have

$$\mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}.$$

It follows from this observation that, for the original problem, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{24}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{9}{s^2+9}\right\}.$$

We have to be careful at this point. We need to recognize that the inverse Laplace transfer forms *depend explicitly on constants*. For instance, we recognize that the first term corresponds structure to the form required for

$$\mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\} = x^{n-1}$$

with $n = 4$. We will have to make sure that, however, that the constant in the numerator is the correct one! If we have $n = 4$, we must have $(n-1)! = 3! = 6$ absorbed by the inverse Laplace transform.

We will also have to be careful with the form

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bx).$$

We recognize that $b = 3$ for our given form, so this much will be absorbed by the inverse transformation. All told, we should recognize that we have

$$\mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = 4x^3 - 3\sin(3x).$$

Example 2: Find the inverse Laplace transform of

$$F(s) = \frac{8}{s^3 + 4s}.$$

Solution: Our first observation is that, no matter how we adjust the constant in the numerator, this does not fit readily into our given forms. So what can we do?

The key—which will be a common feature of inverting Laplace transforms—is that we can factor the denominator into

$$F(x) = \frac{8}{s^3 + 4s} = \frac{8}{s(s^2 + 4)}.$$

This may seem like a modest gain, but it actually leads to a general method. We know from work on integrate that we can break terms like this into *separate* terms with simpler denominators. In this case, we know that we can write this as

$$\frac{8}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

for some constants A , B , and C by using *partial fraction decomposition*. The forms on the right-hand side are easily recognized from your table of inverse transforms! So partial fraction decomposition is the key to resolving inverse transformations with complicated denominators.

In this case, we have

$$8 = A(s^2 + 4) + (Bs + C)s \implies (A + B)s^2 + Cs + (4A - 8) = 0.$$

Equating coefficients on both sides gives the system

$$\begin{aligned} A + B &= 0 \\ C &= 0 \\ 4A - 8 &= 0. \end{aligned}$$

It follows from the final constraint that $A = 2$ so that $B = -2$ by the first. Also, clearly $C = 0$ by the second. It follows that we have

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{8}{s(s^2 + 4)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2}{s} - \frac{2s}{s^2 + 4} \right\} \\ &= 2\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - 2\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} \\ &= 2 - 2\cos(2x). \end{aligned}$$

1.3 Shifted Laplace Transforms

A particular nice property of Laplace transforms is that it very easily handles shifts in the domain variable s . For instance, imagine trying to determine the inverse Laplace transform

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2 + 1} \right\}.$$

While we can certainly identify elements of the Laplace transforms of $\sin(x)$, it is not quite in the necessary form because of the domain shift $s - 1$ in the denominator.

The following result allows us to compute such inverse transformations.

Lemma 1.1 (Theorem 6.3.2 in text). *Suppose that $\mathcal{L}\{f(x)\} = F(s)$. Then $\mathcal{L}\{e^{cx}f(x)\} = F(s-c)$. Conversely, we have $\mathcal{L}^{-1}\{F(s-c)\} = e^{cx}f(x)$.*

While this might seem startling—how do go from a shift in s to an extra exponential form?—the proof is very simple. By definition, we have

$$\mathcal{L}\{e^{cx}f(x)\} = \int_0^\infty e^{-sx}e^{cx}f(x) dx = \int_0^\infty e^{-(s-c)x}f(x) dx = F(s-c).$$

An important consequence of Lemma 1.1 is that it very quickly allows us to expand our lexicon of Laplace transformations. We may now add:

$$\begin{aligned} \mathcal{L}\{e^{ax}x^n\} &= \frac{n!}{(s-a)^{n+1}} & \implies & \mathcal{L}^{-1}\left\{\frac{(n-1)!}{(s-a)^n}\right\} = e^{ax}x^{n-1} \\ \mathcal{L}\{e^{ax}\sin(bx)\} &= \frac{b}{(s-a)^2 + b^2} & \implies & \mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{ax}\sin(bx) \\ \mathcal{L}\{e^{ax}\cos(bx)\} &= \frac{(s-a)}{(s-a)^2 + b^2} & \implies & \mathcal{L}^{-1}\left\{\frac{(s-a)}{(s-a)^2 + b^2}\right\} = e^{ax}\cos(bx) \end{aligned}$$

where $s > a$. For our above example, we may now quickly identify that the inverse transformation gives

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2 + 1} \right\} = e^x \sin(x).$$

Example: Determine the inverse Laplace transform of

$$F(s) = \frac{s-1}{s^2 - 4s + 5}.$$

Solution: We need to determine

$$\mathcal{L}^{-1} \left\{ \frac{s-1}{s^2 - 4s + 5} \right\}$$

but do not recognize this immediately as being in one of our prescribed forms. Worst still, the bottom cannot be factored (over the real numbers, anyway) so that we cannot split the denominator.

Our only alternative is to *complete the square* in the denominator. This will be a general method, in fact. If we have an irreducible quadratic term, we must complete the square to get it in the standard form $(s-a)^2 + b^2$. In this case, we have

$$s^2 - 4s + 5 = (s^2 - 4s + 4) - 4 + 5 = (s-2)^2 + 1.$$

Now we are getting somewhere! We have

$$\mathcal{L}^{-1} \left\{ \frac{s-1}{s^2 - 4s + 5} \right\} = \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-2)^2 + 1} \right\}.$$

This is pretty good, but we are not out of the woods yet. The shifted sine and cosine forms require us to have some factor of either the shift (i.e. $s-c$) or the remaining term (i.e. b) in the numerator. We clearly have $c=2$ and $b=1$ but we do not have $s-2$ or 1 by themselves in the numerator. Instead, we must *create* them. In this case, we can simply adjust to get what we need. If we subtract by a one in the numerator, we need to add by one. This gives

$$\mathcal{L}^{-1} \left\{ \frac{s-1-1+1}{(s-2)^2 + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s-2}{(s-2)^2 + 1} + \frac{1}{(s-2)^2 + 1} \right\}.$$

This is exactly we needed! We can immediately recognize these as the shift sine and cosine forms. After a little bit of work, we have been able to show that

$$\mathcal{L}^{-1} \left\{ \frac{s-1}{s^2 - 4s + 5} \right\} = e^{2x} \cos(x) + e^{2x} \sin(x).$$

1.4 Laplace Transform of Derivatives

It should not be immediately obvious how all the work we have done so far will help us in solving *differential equations*. After all, we have built up a lexicon of Laplace transformations for given functions, but have said nothing at all about derivatives. At this point, we should still be skeptical of our lofty expectations!

Now we will tackle this problem head on. Suppose now that we are asked to take the Laplace transform of a *derivative*, something like

$$\mathcal{L}\{f'(x)\}.$$

How might we handle something like this? After all, we do now know what $f(x)$ is and therefore have no way of knowing what $f'(x)$ is. Context will play a key role here: for differential equations, we may have information about derivatives but *not* about the function $f(x)$ itself. Rather, that is what we want to find. So when taking Laplace transforms, we will be satisfied to simply relate the Laplace transform of $f'(x)$ to the Laplace transform of $f(x)$ itself.

Suppose that $F(s)$ is the Laplace transform of $f(x)$, by the definition of the Laplace transform we have

$$\begin{aligned}\mathcal{L}\{f'(x)\} &= \int_0^{\infty} e^{-sx} f'(x) dx \\ &= \lim_{A \rightarrow \infty} e^{-sA} f(x) \Big|_0^A + s \int_0^{\infty} e^{-sx} f(x) dx \\ &= \lim_{A \rightarrow \infty} e^{-sA} f(A) - f(0) + sF(s)\end{aligned}$$

where

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x) dx.$$

Again, as is becoming a regular trend, we have used integration by parts.

We have to be somewhat careful when evaluating the remaining limit. We would like to say that the limit converges in the limit to zero for $s > 0$, since the term e^{-sA} does, but that will only be true if $f(A)$ does not grow faster than exponential in the limit as $A \rightarrow \infty$ (in math language, that is to say, we required that $|f(x)| \leq ke^{ax}$ where k and a are some constants). That is to say, we could *not* allow something like $f(x) = e^{x^2}$, which grows very, *very* quickly relative to a standard exponential function (in fact, *any* standard exponential function).

We will sweep this technicality aside, since we will typically be dealing with functions like exponentials, polynomials, sines, and cosines. For these functions, we may certainly say the growth is at most exponential, and therefore that

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0).$$

We may, in fact, go quite a bit further. This was just the *first* derivative, and we often have equations with higher-order derivatives as well. It can be checked using the same method that

$$\mathcal{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0).$$

In general, we have that

$$\mathcal{L}\{f^{(n)}(x)\} = s^n F(s) - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

What is interesting about this is that the Laplace transform of a derivative can be expressed in terms of the Laplace transform of the base function *together with* the values of the function at the first point of integral (usually taken to be $x = 0$). In terms of differential equations, this will be a very useful property, since the values at $x = 0$ will correspond to the initial conditions in our differential equation problem.

1.5 Solving Initial Value Problems

We may now (finally!) combine all this groundwork toward our ultimate goal of solving differential equations. We will start with a simple example:

$$y''(x) - 3y'(x) + 2y(x) = 0, \quad y(0) = 0, y'(0) = 1.$$

Since this is linear and has constant coefficients, we may solve this equation by guessing $y(x) = e^{rx}$ and solving for r . After substituting the initial conditions, this would give us that answer

$$y(x) = -e^x + e^{2x}.$$

We now want to solve this by using a Laplace transform method!

Our first step is to take the Laplace transform of the whole equation. We have

$$\begin{aligned} & \mathcal{L}\{y''(x) - 3y'(x) + 2y(x)\} = 0 \\ \implies & \mathcal{L}\{y''(x)\} - 3\mathcal{L}\{y'(x)\} + 2\mathcal{L}\{y(x)\} = 0 \\ \implies & [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = 0 \end{aligned}$$

where $Y(s)$ is the Laplace transform of $y(x)$. What is interesting here is that all of the derivatives have been absorbed into the initial conditions, and we know what these are! The only thing unsolved for here is $Y(s)$. Consequently, we will attempt to isolate this term:

$$\begin{aligned} \implies [s^2Y(s) - 1] - 3sY(s) + 2Y(s) &= 0 \\ \implies (s^2 - 3s + 2)Y(s) &= 1 \\ \implies Y(s) &= \frac{1}{s^2 - 3s + 2}. \end{aligned}$$

This is great! We have now isolate the Laplace transform of the solution we want. It remains only to invert the transformation, and we have already perform this task a few times. We will need to factor and perform partial fraction decomposition on the right-hand side. We have

$$Y(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

We can multiply this across to get

$$1 = A(s-2) + B(s-1).$$

Setting $s = 1$ gives $A = -1$ and setting $s = 2$ gives $B = 1$. It follows that we have

$$Y(s) = -\frac{1}{s-1} + \frac{1}{s-2}.$$

It follows that, as expected, we have

$$y(s) = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = -e^x + e^{2x}.$$

We should be very happy that we obtained the earlier expected answer, but we might wonder *why* we needed another method in the first place. After all, the standard method works just fine for this example. The answer will not be fully apparent just yet, but it is worth noting that the Laplace transform method is a *one-step* solution method. Our previous method required separate steps to hand homogeneities and initial conditions. The Laplace transform method directly incorporates these things into the method itself.

To see how non-homogeneities are incorporated in the method, consider the following problem.

Example: Use the Laplace transform method to solve

$$y''(x) + 4y'(x) + 4y(x) = 3xe^{-2x}, \quad y(0) = 0, y'(0) = 1.$$

Solution: We have

$$\mathcal{L}\{y''(x) + 4y'(x) + 4y(x)\} = \mathcal{L}\{3xe^{-2x}\}$$

$$\implies [s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4Y(s) = \frac{3}{(s+2)^2}$$

$$\implies (s+2)^2Y(s) = \frac{3}{(s+2)^2} + 1$$

$$\implies Y(s) = \frac{s^2 + 4s + 7}{(s+2)^4}$$

We need to perform partial fraction decomposition, so we set up

$$\frac{s^2 + 4s + 7}{(s+2)^4} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}.$$

Multiplying across gives

$$s^2 + 4s + 7 = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$s^2 + 4s + 7 = As^3 + (6A+B)s^2 + (12A+4B+C)s + (8A+4B+2C+D).$$

It follows that we need to satisfy the system

$$A = 0$$

$$6A + B = 1$$

$$12A + 4B + C = 4$$

$$8A + 4B + 2C + D = 7.$$

It follows from the first equation that $A = 0$, so that the second equation gives $B = 1$. The third equation then gives $C = 0$ so the fourth gives $D = 3$. It follows that we have

$$\begin{aligned} Y(s) &= \mathcal{L}^{-1}\left\{\frac{s^2 + 4s + 7}{(s+2)^4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2} + \frac{3}{(s+2)^4}\right\} \end{aligned}$$

We will have to be careful with the second form here. We recognize it as a shifted form of x^3 but the required form in the numerator is $3! = 6$, not 3. We have

$$Y(s) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} = xe^{-2x} + \frac{x^3}{2}e^{-2x}.$$

This form can be easily verified directly or by solving using the classical method, but requires separate steps to determine the complementary solution $y_c(x)$, the particular solution $y_p(x)$, and then to evaluate the initial conditions. By contrast, the Laplace transform method employed above was a *one-step* method.