MATH 319, WEEK 11 & 12: Laplace Transforms: Discontinuous Forcing Functions

1 Piecewise Functions

In the motivation for Laplace Transforms, we were led to believe that one of the primary advantages of this method is that it easily handles *non-smooth* and even *discontinuous* forcing functions (i.e. nonhomogeneities). In order to handle such cases, we must expend a little energy developing the framework for the Laplace transform of *piecewise-defined functions*.

Consider computing the Laplace transform of the following function:

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & x \geq 2 \end{cases} \tag{1}$$

Such a function is commonly called a "tent" function (see Figure 1).

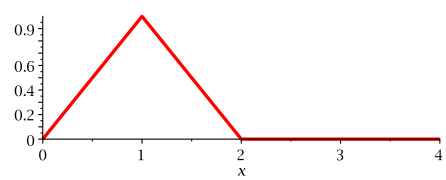

Figure 1: The piecewise defined function $g(t)$ given in equation (1) over the interval $0 \leq x \leq 4$.

This could model, for instance, an external signal that begins to climb at $x = 0$, then begins to fall at $x = 1$, and eventually reaches zero (i.e. no signal) at $x = 2$. Such forcing functions were notoriously difficult to handle in the classical differential equation setting since we essentially had to solve the differential equation independently in each region—i.e. we had to solve the differential equation *three* times!

Now consider computing the Laplace transform of such a function. From the definition, we have that

$$\begin{aligned}
\mathcal{L}\left\{f(x)\right\} &= \int_0^\infty e^{-sx} f(x)\, dx \\
&= \int_0^1 xe^{-sx}\, dx + \int_1^2 (2-x)e^{-sx}\, dx \\
&= \left[-\frac{e^{-s}}{s} + \frac{1}{s}\int_0^1 e^{-sx}\, dx\right] - 2\frac{e^{-2s}}{s} + 2\frac{e^{-s}}{s} \\
&\quad + \left[2\frac{e^{-2s}}{s} - \frac{e^{-s}}{s} - \frac{1}{s}\int_1^2 e^{-sx}\, dx\right] \\
&= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right)
\end{aligned}$$

where we have applied integration by parts several times and cleaned up the resulting expression.

That was a fair bit of work, but we should be relatively happy with the outcome! What this tells us that, in the Laplace transform world, piecewise defined functions correspond to a *single* function of s. This is a big deal! We will be able to solve differential equations with piecewise-defined forcing terms in *exactly the same way* as we have been traditional forcing terms (or no forcing term at all).

2 Heaviside Functions

It is not convenient to apply the definition of a Laplace transform every time we use one. Rather, we want to develop a system of rules for handling piecewise defined functions. The key to accomplishing this will be first defining the *Heaviside function*

$$u_c(x) = \left\{ \begin{array}{ll} 0, & 0 \leq x < c \\ 1, & x \geq c \end{array} \right.$$

The Heaviside function can be thought of as an "on"/"off" switch with a trigger value c. If we look to the left of c, the function evaluates to zero (the "off" state), and if we look to the right of c, the function evaluates to one (the "on" state).

The importance of the Heaviside function lies in the fact that it can be combined with itself and other functions to generalize the notion of turning *functions* "on" or "off" over certain regions of x. In particular, if $d > c$ we can define

$$u_c(x) - u_d(x) = \begin{cases} 0, & 0 \leq x < c \\ 1, & c \leq x < d \\ 0, & x \geq d \end{cases}$$

In other words, we are only in the "on" state in the region $c \leq x < d$; otherwise, we are "off". So this form allows us to define *bounded* intervals which are "on".

Of course, what we are interested in turning "off" and "on" is not simply the value one. Rather, we are manipulating *functions*. In particular, consider the piecewise define function defined earlier:

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & x \geq 2 \end{cases}$$

What this definition means is that the function $f_1(x) = x$ is "on" in the region $0 \leq x < 1$, and then turned "off" at $x = 1$ when the new function $f_2(x) = 2 - x$ is turned "on". Finally, at $x = 2$, $f_2(x) = 2 - x$ is turned "off" and the trivial function $f_3(x) = 0$ is turned "on".

In fact, we can make use of exactly this intuition! For $f_1(x) = x$ to be turned "on" in the region $0 \leq x < 1$, we need to have

$$(1 - u_1(x))f_1(x) = (1 - u_1(x))x$$

where we notice that $1 - u_1(x)$ is "on" for $0 \leq x < 1$ and "off" for $x \geq 1$. Similarly, the idea of turning $f_2(x) = 2 - x$ "on" at $x = 1$ and "off" at $x = 2$ is captured by

$$(u_1(x) - u_2(x))f_2(x) = (u_1(x) - u_2(x))(2 - x).$$

Finally, we can turn $f_3(x) = 0$ "on" at $x = 2$ with

$$u_2(x)f_3(x) = 0.$$

It follows that the piecewise defined function can be written in terms of Heaviside functions as

$$\begin{aligned} f(x) &= (1 - u_1(x))x + (u_1(x) - u_2(x))(2 - x) \\ &= x + 2u_1(x)(1 - x) + u_2(x)(x - 2) \end{aligned}$$

3 Laplace Transform of Heaviside Functions

We have already seen that we could compute the Laplace transform of piecewise defined functions, so let's see how the Laplace transforms handles the Heaviside function. First of all, by the definition we can see that

$$\mathcal{L}\{u_c(x)\} = \int_0^\infty u_c(x)e^{-sx}\,dx = \lim_{A\to\infty}\int_c^A e^{-sx} = \frac{e^{-cs}}{s}, \quad s > 0.$$

In particular, we notice that this generalizes for the case $c = 0$, corresponding to a function which is always "on", to the identity

$$\mathcal{L}\{u_0(x)\} = \mathcal{L}\{1\} = \frac{1}{s}.$$

Whenever we see a term e^{-cs} in the transformed world, therefore, we will immediately suspect that the Heaviside function is involved. Notice that we also have the inverse identity

$$\mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\} = u_c(x).$$

We now want to consider what happens to *functions* which are turned "off" or "on" at a particular value. We know that we can formulate this intuition using Heaviside functions, so this is really a question of how we take Laplace transforms of functions which *interact* with Heaviside functions. We have the following result.

Theorem 3.1. *Suppose $F(s) = \mathcal{L}\{f(x)\}$ and $u_c(x)$ is the Heaviside function centered at $c \geq 0$. Then*

$$\mathcal{L}\{u_c(x)f(x-c)\} = e^{-cs}F(s)$$

and, conversely,

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u_c(x)f(x-c).$$

What is distinctive about this result is that the domain shift $x - c$ in the y and x variable system disappears when we apply the transformation. While this is perhaps unexpected—and will be easy to forget!—the proof is straight-forward. By definition, we have

$$\begin{aligned}
\mathcal{L}\left\{u_c(x)f(x-c)\right\} &= \int_0^\infty u_c(x)e^{-sx}f(x-c)\,dx \\
&= \int_c^\infty e^{-sx}f(x-c)\,dx \\
&= e^{-cs}\int_0^\infty e^{-s\tilde{x}}f(\tilde{x})\,d\tilde{x} \\
&= e^{-cs}F(s)
\end{aligned}$$

where we have made the substitution $\tilde{x} = x - c$ (so that $dx = d\tilde{x}$ and $e^{-sx} = e^{-sc}e^{-s\tilde{x}}$).

Note: The trick with applying this result will be to make sure that the function multiplying the Heaviside function is *always* arranged in factors of $x - c$. Otherwise, the result does not apply and our answer will be wrong!

Example 1: Determine the Laplace transform of $x^2u_1(x)$.

Solution: We want to use our Theorem, but we cannot directly evaluate

$$\mathcal{L}\left\{x^2u_1(x)\right\}$$

because $f(x) = x^2$ is not factored according to $x - 1$. This can be corrected by adding and subtracting terms appropriately. In this case, we notice that we have

$$(x-1)^2 = x^2 - 2x + 1.$$

Rearranging, we have

$$x^2 = (x-1)^2 + 2x - 1 = (x-1)^2 + 2(x-1) + 1$$

where we had added and subtracted terms as appropriate. Finally, we have

$$\begin{aligned}
\mathcal{L}\left\{x^2u_1(x)\right\} &= \mathcal{L}\left\{((x-1)^2 + 2(x-1) + 1)u_1(x)\right\} \\
&= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).
\end{aligned}$$

Notice that in the last step we have ignored the time-shift! This is because, making the substitution $\tilde{x} = x - 1$, we have $f(x-1) = f(\tilde{x}) = \tilde{x}^2 + 2\tilde{x} + 1$. *This* is the function corresponding to the Laplace transform $F(s)$ in the

Theorem!

Example 2: Determine the inverse Laplace transform of

$$e^{-\pi s}\frac{4}{s^2+16}.$$

Solution: We have the strong indication from the $e^{-\pi s}$ in the transform that there will be a Heaviside function $u_\pi(x)$ in our solution. In particular, we expect the shift $x-\pi$. First of all, however, we recognize that

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \sin(4x).$$

Applying our shift $x-\pi$ to this form, we have that

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{4}{s^2+16}\right\} = u_\pi(x)\sin(4(x-\pi)).$$

Example 3: Use Theorem 3.1 to determine the Laplace transform of

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & x \ge 2 \end{cases}$$

Solution: We know from our earlier work that $f(x)$ can be written in the form

$$f(x) = x + 2u_1(x)(1-x) + u_2(x)(x-2).$$

In order to determine the Laplace transform, we need to compute

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{x\} + 2\mathcal{L}\{u_1(x)(1-x)\} + \mathcal{L}\{u_2(x)(x-2)\}.$$

The only trick at this point is that we need each term multiplying a Heaviside function $u_c(x)$ to be expressed in terms of the difference $x-c$. In this case, we are almost done! We already have the differences $x-1$ and $x-2$ explicitly in the equations (this is not generally the case!). We may choose one final piece of simplification by get

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \mathcal{L}\{x\} - 2\mathcal{L}\{u_1(x)(x-1)\} + \mathcal{L}\{u_2(x)(x-2)\} \\ &= \frac{1}{s^2}\left(1-2e^{-s}+e^{-2s}\right)\end{aligned}$$

as before.

4 Piecewise-Defined Forced Initial Value Problems

Consider now solving the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x(t) = g(t), \quad x(0) = 0, x'(0) = 0$$

where

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases} \tag{2}$$

This could correspond to the model for a damped pendulum which undergoes an escalation in forcing to the right which grows linearly until $t = 1$, and then starts to ease before disappearing entirely at $t = 2$ (an escalating breeze, perhaps). Doing this by our previous method would requite solving the differential equation *three* times in each of the intevals. Now we want to determine the solution using the (one-step) Laplace transform method.

First of all, we have to take the Laplace transform of the entire differential equation—including the piecewise-defined forcing term. Although we recognize this as the function we have already dealt with, it is important to stress the steps we need to get. We need to first rewrite the piecewise-defined function $g(t)$ in terms of Heaviside functions. We have

$$\begin{aligned} g(t) &= (1 - u_1(t))t + (u_1(t) - u_2(t))(2 - t) \\ &= t + 2u_1(t)(1 - t) + u_2(t)(t - 2). \end{aligned}$$

The equation we must take the Laplace transform of, therefore, is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x(t) = t + 2u_1(t)(1 - t) + u_2(t)(t - 2).$$

The Laplace transform gives

$$[s^2X(s) - sx(0) - x'(0)] + 2[sX(s) - x(0)] + 2X(s) = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right)$$

$$\Longrightarrow (s^2 + 2s + 2)X(s) = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right)$$

$$\Longrightarrow X(s) = \frac{1}{s^2(s^2 + 2s + 2)}\left(1 - 2e^{-s} + e^{-2s}\right).$$

We can break this solution into smaller parts so that $X(s) = X_1(s) + X_2(s) + X_3(s)$ where

$$\begin{aligned} X_1(s) &= H(s) \\ X_2(s) &= -2H(s)e^{-s} \\ X_3(s) &= H(s)e^{-2s} \end{aligned}$$

and

$$H(s) = \frac{1}{s^2(s^2 + 2s^2 + 2)}.$$

We now want to invert the transformation to get a solution written in terms of Heaviside functions.

Written as above, we recognize that each portion of the solution depends upon the function $H(s)$ (and eventually, it's inverse Laplace transform form) so we will need to perform partial fraction decomposition. We have

$$\frac{1}{s^2(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+2}$$

$$\begin{aligned} &\Longrightarrow 1 = As(s^2+2s+2) + B(s^2+2s+2) + (Cs+D)s^2 \\ &\Longrightarrow 1 = (A+C)s^3 + (2A+B+D)s^2 + (2A+2B)s + 2B. \end{aligned}$$

This gives rise to the system of equations

$$\begin{aligned} A + C &= 0 \\ 2A + B + D &= 0 \\ 2A + 2B &= 0 \\ 2B &= 1 \end{aligned}$$

which can be solved one variable at a time to get $A = -1/2$, $B = 1/2$, $C = 1/2$, and $D = 1/2$. It follows that we have

$$H(s) = \frac{1}{2}\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{s+1}{s^2+2s+2}\right).$$

Notice now that Theorem 3.1 implies that $\mathcal{L}^{-1}\{X_1(s)\}$, $\mathcal{L}^{-1}\{X_2(s)\}$, and $\mathcal{L}^{-1}\{X_3(s)\}$ all depend upon $\mathcal{L}^{-1}\{H(s)\}$ but with a shift in t. At any rate, we would like to evaluate $\mathcal{L}^{-1}\{H(s)\}$, which requires completing the square in the denominator. We have

$$s^2 + 2s + 2 = s^2 + 2s + 1 + 1 = (s+1)^2 + 1.$$

It follows that we have

$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{s+1}{s^2+2s+2}\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{2}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{s+1}{(s+1)^2+1}\right)\right\} \\
&= \frac{1}{2}\left(-1+t+e^{-t}\cos(t)\right).
\end{aligned}
$$

Returning to our original equation now, we have

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{X(s)\right\} \\
&= \mathcal{L}^{-1}\left\{X_1(s)\right\} + \mathcal{L}^{-1}\left\{X_2(s)\right\} + \mathcal{L}^{-1}\left\{X_3(s)\right\} \\
&= \mathcal{L}^{-1}\left\{H(s)\right\} - 2\mathcal{L}^{-1}\left\{H(s)e^{-s}\right\} + \mathcal{L}^{-1}\left\{H(s)e^{-2s}\right\} \\
&= h(t) - 2u_1(t)h(t-1) + u_2(t)h(t-2)
\end{aligned}
\tag{3}
$$

where $h(t)$ is as above.

It is not easy to see from this form exactly what is happening, but computer software packages make it extremely transparent (see Figure 2 and 3). We can see that the solution is initially at rest but is eventually set in motion by the escalating forcing term. The forcing term begins descending at $t = 1$, just as the solution is gaining some speed. Once the forcing term is removed at time $t = 2$, the now displaced pendulum/spring is free to settle back into its natural rhythm. In this case, it will settle into damped oscillations, since the unforce system has the solution form

$$x(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t).$$

To make the example more practical, we could imagine a pendulum hanging in its equilibrium position which is suddenly displaced by an escalating gust of wind. The wind will blow the pendulum toward the side, picking up speed along the way, until the air is still again and the pendulum is free to swing back to its equilibrium position.

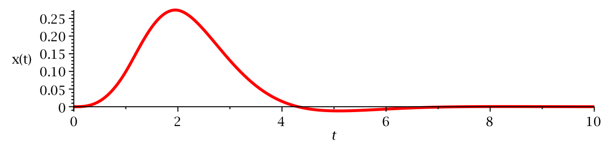

Figure 2: Plot of the solution $x(t)$ given by (3).

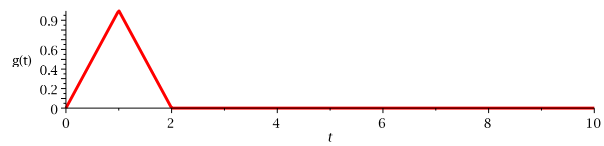

Figure 3: Plot of the piecewise-defined forcing function $g(t)$ given by (2). Notice that the solution picks up speed when the forcing is ascending ($0 \leq t < 1$) and then slows down when the forcing is scaled back ($1 \leq t < 2$). When the forcing is no longer present ($t \geq 0$), the solution returns to its equilibrium position.