

MATH 319, WEEK 12: Crash Course in Linear Algebra

1 Linear Algebra

Throughout the remainder of the course, we will be dealing with *systems* of first-order differential equations. But how do we represent these equations? Do we *really* have to write them out individually each time we want to refer to them? That seems absurd, and yet it is all we have at our disposal at the moment. What else can we do?

The answer comes from the discipline of *linear algebra*. What we need to do is define a few new objects, called *matrices* and *vectors*, and describe how they interact. Buried inside these objects will be the individual quantities (e.g. variables, functions, derivatives, etc.) which we have been interested in up to this point in the course.

The advantage of briefly introducing linear algebra now is two-fold:

1. It will allow us to condense expressions in a way which will make a first-order system of differential equations analogous to the first-order differential equations studied in the first 5-6 weeks of the course.
2. Theoretical results from linear algebra will be *essential* in deriving solutions to all but the simplest of such systems.

That said, linear algebra is not a primary topic of Math 319. We will introduce only the topics which are necessary for our study, and for theoretical and computational simplicity, will limit ourselves to two-dimensional systems. (The interested student is directed to Math 320 and/or Math 340 for a fuller introduction.)

2 Matrices and Vectors

The basic objects of study in linear algebra are *matrices* and *vectors*, which we will (usually) consider to be over the real numbers \mathbb{R} .

Definition 2.1. A *matrix* $A \in \mathbb{R}^{m \times n}$ is defined to be

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} \in \mathbb{R}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. A *vector* $\mathbf{v} \in \mathbb{R}^m$ is defined to be

$$\mathbf{v} = (v_1, v_2, \dots, v_m)$$

where $v_i \in \mathbb{R}$ for all $i = 1, \dots, m$.

There are a few notes worth making about these objects:

- A matrix can be thought of as a rectangular *grid* of numbers. For example, we might have

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & \frac{1}{2} & 6 & -2 \end{bmatrix}$$

or

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- It is important to note that, for each matrix entry a_{ij} , the first index i corresponds to the *row* while the second index j corresponds to the *column*. This is standard across all disciplines which use linear algebra (which is a lot!).
- Matrices will be denoted with capital letters (usually) in alphabetical order. That is to say, if we have multiple matrices in a problem, we will denote them by A , B , C , and so on.
- Vectors will alternatively be denoted by bold-faced (e.g. \mathbf{v}) or with an arrow overtop (e.g. \vec{v}). Bold-faced will be favored in the online notes while the arrow notation will be favored in class.
- A vector can be thought of as a matrix with either one row (a *row vector*) or one column (a *column vector*). For example, if we have

$$\mathbf{v} = [2 \ 3], \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then \mathbf{v} is a row vector, and \mathbf{w} is a column vector.

- Addition and scalar multiplication of matrices (and vectors) is defined component-wise. That is to say, if we have

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

then we have

$$\begin{aligned} A + 2B &= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2(0) & 2 + 2(-1) \\ -3 + 2(1) & 1 + 2(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}. \end{aligned}$$

3 Multiplication

We will also need to define multiplication for matrices. This is the first operation which is not as intuitive as applying the standard operation component-wise to the relevant matrices. Rather, we have the following definition.

Definition 3.1. *Suppose $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$. Then the matrix $C = AB \in \mathbb{R}^{m \times n}$ is defined to be the matrix with entries*

$$c_{ij} = \sum_{k=1}^p a_{ik}b_{kj}.$$

Alternatively, we may think of matrix multiplication as consisting of multiplying rows of the first matrix component-wise with columns of the second, and then adding the results. The result of this operation is then placed in the row and column of the new matrix corresponding to the row and column which were just multiplied together. For example, for the A and B just defined, we have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (1)(0) + (2)(1) & (1)(-1) + (2)(2) \\ (-3)(0) + (1)(1) & (-3)(-1) + (1)(2) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}. \end{aligned}$$

Note: There are two technicalities that arise out of this definition of matrix multiplication:

1. The operation is only defined if the first matrix has *exactly* the same number of columns as the second matrix has rows. For many matrices the product AB is not even defined!
2. Even if the operation is defined, it is not generally the case that $AB = BA$ (check with the previous example!). In other words, order matters. This is a significant distinction with multiplication of real numbers where $x \cdot y = y \cdot x$ trivially, and will be easy to forget.

4 2×2 Inverses

A particularly important matrix is the *identity matrix*, which for 2×2 systems is defined as $I \in \mathbb{R}^{2 \times 2}$ where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix has the property that $AI = A$ and $IA = A$ for any matrix for which the multiplication is defined. Consequently, in the theory of linear algebra, the identity matrix can be thought of as being analogous to the value one in standard algebra.

An interesting question is whether, given a square matrix A , we can find a matrix which multiplies to give the identity matrix. That is to say, can we find a matrix B such that

$$AB = I.$$

There are numerous reasons we might want to do this, but we will not go into any significant detail here. (We may think of it as defining *division* for matrices, since it will allow us to move matrices from one side of an equation to the other.)

The following matrix accomplishes this task.

Definition 4.1. Consider a matrix $A \in \mathbb{R}^{2 \times 2}$ defined by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the *inverse* matrix $A^{-1} \in \mathbb{R}^{2 \times 2}$ is defined by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and has the property that $A^{-1}A = AA^{-1} = I$.

The 2×2 inverse matrix A^{-1} has many nice properties. Most notably, it is guaranteed to be an inverse on the right *and* on the left ($A^{-1}A = I$ implies $AA^{-1} = I$, and vice versa), and it is unique (i.e. given a matrix A there exactly one matrix satisfying $A^{-1}A = AA^{-1} = I$). That the given matrix satisfies $A^{-1}A = I$ and $AA^{-1} = I$ can be verified directly.

An immediate consequence of this formula is that not every matrix has an inverse (i.e. not all matrices are *invertible*). This follows from the fact that $ad - bc$ may be zero, in which case the formula tells us to divide by zero. (This could have been anticipated by our interpretation of inverses as the matrix analogue of division!) This quantity is important enough in the theory of linear algebra that it is given its own name.

Definition 4.2. The **determinant** of a matrix $A \in \mathbb{R}^{2 \times 2}$ is defined by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Many properties follow from the value of the determinant, but the one which is more readily apparent (and frequently used!) is that a matrix is invertible *if and only if* $\det(A) \neq 0$.

Example: Determine whether the following matrices are invertible and, if so, find the inverse:

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

Solution: We can simply apply the formula. We first need to compute the determinants. We have

$$\det(A) = 2(-3) - (-5)(1) = -6 + 5 = -1$$

and

$$\det(B) = 2(2) - (-1)(-4) = 4 - 4 = 0.$$

It follows that A is invertible and B is not. The inverse of A , A^{-1} can be computed by the formula as

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}.$$

We can easily verify this is correct by computing

$$A^{-1}A = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

5 Eigenvalues and Eigenvectors

We now introduce the following a pair of objects which will be the primary focus of our application of linear algebra to systems of differential equations.

Definition 5.1. Consider a matrix $A \in \mathbb{R}^{2 \times 2}$. Then a vector $\mathbf{v} \in \mathbb{C}^2$ satisfying the equation

$$A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

for some $\lambda \in \mathbb{C}$ is called an **eigenvector** of A , and the value λ is the corresponding **eigenvalue**.

What is important to note about equation (1) is that the relationship $A\mathbf{v}$ always produces a vector (i.e. a 2×1 matrix) but that this vector usually has *nothing* very much to do with the original vector \mathbf{v} . That is to say, if we picked a 2×2 matrix and a vector \mathbf{v} at random, we would not expect to see much similarity between \mathbf{v} and the vector produced by $A\mathbf{v}$. For instance, if we pick the matrix A and vector \mathbf{v} to be

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

we compute

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

As we expected, this has produced another vector, but the vector it has produced has very little resemblance to the vector we started with, $\mathbf{v} = (1, 1)$.

It is sometimes the case, however, for very carefully selected vectors, computing $A\mathbf{v}$ produces a vector which differs from \mathbf{v} by only a constant value (a scaling in the (x, y) -plane). In some sense, the operation implied by A on the vector \mathbf{v} does not change the affect the vector. Such a pair is called an eigenvalue/eigenvector pair. For example, if we choose the vector $\mathbf{v} = (1, 2)$ with the above matrix A , we have

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We will say that $\mathbf{v} = (1, 2)$ is an eigenvector of A with eigenvalue $\lambda = 2$.

This property may not seem like a big deal, but it turns out that, for many applications, determining the set of eigenvectors \mathbf{v} and corresponding eigenvalues λ of a matrix A is sufficient to tell us everything we need to

know about what A does, has ever done, and will ever do. These vectors are the key to unlocking the behavior of matrices in a wide various contexts. (The full extent of these details will unfortunately not be presented in this course. A more detailed treatment is given in Math 340.)

The question then becomes: How do we *find* eigenvalues and eigenvectors? How did I *know* to choose the vector $\mathbf{v} = (1, 2)$? Was it just a lucky guess? If they are so important, we should be able to do better than just guess what they are!

It is fortunate that there is a way to determine the eigenvalues and eigenvectors of any matrix. The argument, however, depends slightly on linear algebra that is beyond what has been presented in this write-up. We should nevertheless be able to apply the steps. What we need to realize is that we can write the eigenvector equation (1) as

$$A\mathbf{v} = \lambda I\mathbf{v} \implies A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}. \quad (2)$$

We are allowed to do this because the identity matrix does not alter a matrix (or vector) upon multiplication so that $\mathbf{v} = I\mathbf{v}$.

The resulting matrix on the left-hand side is $A - \lambda I$, which is the matrix A with λ subtracted from the diagonal. It is a basic fact of linear algebra (but unfortunately not one we will be able to justify in this course) that a matrix can satisfy the equation $A\mathbf{v} = \mathbf{0}$ for a non-trivial vector \mathbf{v} if and only if its determinant is zero. That is to say, the equation (2) can be true if and only if

$$\det(A - \lambda I) = 0. \quad (3)$$

So far, all the arguments we have made apply to a *general* square matrix A . Now consider the general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can now clearly see that

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}.$$

It follows that $\det(A - \lambda I) = 0$ implies

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = 0. \quad (4)$$

This equation is called the *characteristic polynomial* of the matrix A . It is easy to generalize to higher dimensions, but the form of the equation is more difficult to write down in general.

What is important to note is that (4) is a quadratic expression in the eigenvalues λ . It follows that we can determine the eigenvalues of the matrix by solving this quadratic! If we wanted to be explicit about it, we could write

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

It will generally be easier to solve this quadratic on a case-by-case basis than remember this formula.

It is worth noticing, however, that this expression tells us a great deal about the general nature of eigenvalues (and eigenvectors by association). In particular, a 2×2 matrix may only have two eigenvalues, and there are three distinct cases they may fall into, which depends on the sign of $(a + d)^2 - 4(ad - bc)$:

1. There are two real-valued and distinct eigenvalues (say, λ_1 and λ_2).
2. There is one repeated real-valued eigenvalue (say, λ).
3. There is a complex-valued conjugate pair of eigenvalues (say, $\lambda_{1,2} = \alpha \pm \beta i$).

Given an eigenvalue, it remains to find the associated eigenvector $\mathbf{v} = (v_1, v_2)$. In this case, we need only return to (2). Now that we have the eigenvalue λ , we can solve the expression

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

for the vector \mathbf{v} ! The details of this operation are perhaps best left to an examples. (The required actions will be slightly different depending on the which case we are in for the eigenvalues.)

Example 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}.$$

Solution: To compute the eigenvalues, we need to solve

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & 0 \\ 2 & 1 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(1 - \lambda) - (2)(0) \\ &= (2 - \lambda)(1 - \lambda) = 0. \end{aligned}$$

It is clear that we have two distinct real-valued roots, $\lambda_1 = 1$ and $\lambda_2 = 2$. Substituting the first value in the expression $(A - I)\mathbf{v} = \mathbf{0}$ to find the corresponding eigenvector gives

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where we are trying to solve for v_1 and v_2 . An expression like this is called a *linear system of equations*. They are commonly rewritten in the form

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right].$$

The question becomes how to solve such systems of equations. It turns (details omitted) that we are allowed perform the following three operations without changing the value of the solution, v_1 and v_2 :

1. Multiply rows by scalar values.
2. Add rows.
3. Switch the order of rows.

What we want to do is use these operations to rearrange this matrix expression into a form satisfying the following two objectives:

1. The first non-zero entry in each row is a 1.
2. The first non-zero entry in each successive row is further to the right than the previous row.

Such a matrix is said to be in *echelon form*.

For our example (and the vast majority we will see), the method for obtaining this form will be obvious. In this case, we are almost there, since the first row begins with a 1. The only thing we need to worry about is the 2 in the second row. We need to remove it, because we are not permit to have any entry in that column by condition 2 above. We can remove it by taking the second row and subtracting two of the first row from it. We have

$$\tilde{\mathcal{R}}_2 \xrightarrow{\mathcal{R}_2 - 2\mathcal{R}_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

There are no further operations to perform. We recognize now that this corresponds to the system of equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which can be expanded into

$$\begin{aligned}(1)v_1 + (0)v_2 = 0 &\implies v_1 = 0 \\ (0)v_1 + (0)v_2 = 0 &\implies 0 = 0.\end{aligned}$$

The second equation is trivially satisfied while the first places a restriction on v_1 . It follows that the system can be satisfied for any choice of $v_1 = 0$ and $v_2 = t$ where $t \in \mathbb{R}$. That is to say, the vectors we are looking for are

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Choosing $t = 1$ (a convenient choice, but any value will do), we obtain $\mathbf{v}_1 = (0, 1)$.

Now consider $\lambda_2 = 2$. We have

$$(A - 2I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We put this in the condensed matrix form and solve to get

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & -1 & 0 \end{array} \right] \xrightarrow{\tilde{\mathcal{R}}_1 = (1/2)\mathcal{R}_2} \left[\begin{array}{cc|c} 1 & -(1/2) & 0 \\ 0 & 0 & 0 \end{array} \right].$$

It follows that we have the system

$$\begin{aligned}v_1 - (1/2)v_2 &= 0 \\ 0 &= 0.\end{aligned}$$

The second equation (again) is meaningless. This will be a general property of the eigenvector system we will have to solve. This can be satisfied if and only if $v_1 = (1/2)v_2$ which can be parameterized as $v_1 = (1/2)t$ and $v_2 = t$ for $t \in \mathbb{R}$. It follows that we have

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Choosing the value $t = 2$ (an easy choice), we obtain the vector $\mathbf{v}_2 = (1, 2)$. This corresponds to the earlier eigenvector which we confirmed was associated with the eigenvalue $\lambda = 2$ by direct computation, and we are done!

Example 2: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -1 \\ 4 & 0 \end{bmatrix}.$$

Solution: We have

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} -4 - \lambda & -1 \\ 4 & -\lambda \end{vmatrix} \\ &= (-4 - \lambda)(-\lambda) + 4 \\ &= \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.\end{aligned}$$

We do not even need to use the quadratic formula to determine that $\lambda = -2$. We notice that there is something distinctively different about this example since we have only found one eigenvalue instead of the typical two. Nevertheless, we can proceed as before and compute

$$\begin{aligned}(A - 2I) = \mathbf{0} &\implies \left[\begin{array}{cc|c} -4 - (-2) & -1 & 0 \\ 4 & -(-2) & 0 \end{array} \right] \\ &\longrightarrow \left[\begin{array}{cc|c} -2 & -1 & 0 \\ 4 & 2 & 0 \end{array} \right] \\ &\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right].\end{aligned}$$

This corresponds to $v_1 + (1/2)v_2 = 0$. Setting $v_2 = t = 2$ (to clear the denominator) gives us the eigenvector $\mathbf{v} = (-1, 2)$.

Something is very different about this example. We have obtained an eigenvector, but only one, and since there are no further eigenvalues to use, this appears to be the end of the discussion. For the applications involving differential equations, however, we will *need* to have a full set of eigenvectors—in this case, we will need two of them. But if we are not looking for eigenvectors as we have defined them, what *are* we looking for?

The answer is contained in the following definition.

Definition 5.2. A vector $\mathbf{w} \in \mathbb{R}^2$ is called a **generalized eigenvector** of $A \in \mathbb{R}^{2 \times 2}$ if it satisfies

$$(A - \lambda I)^2 \mathbf{w} = \mathbf{0} \text{ but } (A - \lambda I) \mathbf{w} \neq \mathbf{0}.$$

It is not immediately obvious how this new vector helps us (and the rigorous justification is beyond the scope of the course), but it is known that generalized eigenvectors only exist (and exist in exactly the right number!) when there is a *repeated* eigenvalue $\lambda \in \mathbb{R}$. It can also be checked that this equation is equivalent to the far more useful

$$(A - \lambda I) \mathbf{w} = \mathbf{v} \tag{5}$$

where $\mathbf{v} \in \mathbb{R}^2$ is the standard eigenvector corresponding to the eigenvalue $\lambda \in \mathbb{R}$.

Returning to our example, we want to find a generalized eigenvector \mathbf{w} by using the equation $(A - \lambda I)\mathbf{w} = \mathbf{v}$ where $\mathbf{v} = (-1, 2)$. We have

$$\begin{aligned} (A - 2I)\mathbf{w} = \mathbf{v} &\implies \left[\begin{array}{cc|c} -2 & -1 & -1 \\ 4 & 2 & 2 \end{array} \right] \\ &\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

This corresponds to the system $w_1 + (1/2)w_2 = (1/2)$. Setting $w_2 = t$ gives $w_1 = (1/2) - (1/2)t$. Picking $t = 1$ (to clear the denominators) gives $\mathbf{w} = (w_1, w_2) = (0, 1)$. So, corresponding to the repeated eigenvalue $\lambda = 2$, we have the regular eigenvector $\mathbf{v} = (-1, 2)$ and the generalized eigenvector $\mathbf{w} = (0, 1)$.

Example 3: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}.$$

Solution: We need to compute

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & -1 \\ 5 & -1 - \lambda \end{vmatrix} \\ &= (3 - \lambda)(-1 - \lambda) + 5 \\ &= \lambda^2 - 2\lambda + 2 = 0 \\ \implies \lambda &= \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i. \end{aligned}$$

This is quite a bit different than what we have seen so far, but the same principles will apply for complex-valued eigenvalues. We now want to use the formula $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find complex-value eigenvectors.

It turns out that, for complex conjugate eigenvalues, we may pick either sign in the eigenvalue to determine the eigenvector, and the eigenvector corresponding to the eigenvalue with the other sign will be the conjugate of the eigenvector found. This will save some work (but at the cost of having to deal with complex values)!

For $\lambda = 1 + i$ we have

$$\begin{aligned}
 (A - (1 + i)I) \mathbf{v} = \mathbf{0} &\implies \left[\begin{array}{cc|c} 3 - (1 + i) & -1 & 0 \\ 5 & -1 - (1 + i) & 0 \end{array} \right] \\
 &\longrightarrow \left[\begin{array}{cc|c} 2 - i & -1 & 0 \\ 5 & -2 - i & 0 \end{array} \right] \\
 &\longrightarrow \left[\begin{array}{cc|c} (2 - i)(2 + i) & -(2 + i) & 0 \\ 5 & -2 - i & 0 \end{array} \right] \\
 &\longrightarrow \left[\begin{array}{cc|c} 5 & -2 - i & 0 \\ 5 & -2 - i & 0 \end{array} \right] \\
 &\longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{1}{5}i & 0 \\ 0 & 0 & 0 \end{array} \right]
 \end{aligned}$$

This process may seem a little mysterious at first glance, but it will become very familiar—it will work for every 2×2 matrix with complex eigenvalues! What we want to recognize is that:

1. The eigenvector equation guarantees that we *absolutely must* be able to row reduce the condensed matrix so that there is a row of zeroes. That means there is *necessarily* some combination of the two rows which adds to zero. (That combination, however, involves complex numbers.)
2. Multiplying a complex number by its conjugate (e.g. for $\lambda = \alpha + \beta i$, the value $\bar{\lambda} = \alpha - \beta i$ is the conjugate) always produces a real number.

It follows that we just need to match up the real and complex numbers in the columns of the condensed matrix, which is what we have done above. The rest follows as if we had a real-valued matrix!

The final equation is equivalent to $v_1 - ((2/5) + (1/5)i)v_2 = 0$. Setting $v_2 = t$, we have $v_1 = ((2/5) + (1/5)i)t$. Writing in vector form, and then taking $t = 5$, it follows that we have the complex eigenvector

$$\mathbf{v} = \begin{bmatrix} 2 + i \\ 5 \end{bmatrix} = \left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot i \right).$$

The importance of what we have done is that we have obtained two vectors, corresponding to the real and imagine parts of \mathbf{v} —specifically, $\mathbf{a} = (2, 5)$ and $\mathbf{b} = (1, 0)$. These are the two distinct vectors we will need when we return to considering differential equations!

It can be checked that choosing the eigenvalue $\lambda = 1 - i$ would have received the complex eigenvector

$$\mathbf{v} = \left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot i \right).$$

In particular, notice that we have the same real-valued vectors we identified before— $\mathbf{a} = (2, 5)$ and $\mathbf{b} = (1, 0)$.