

# MATH 319, WEEKS 13 & 14: First-Order Systems of Differential Equations

## 1 Motivating Example

A few months ago, we imagined a mathematical model for the chemical reaction  $X \rightarrow Y$  subject to continuous inflow and outflow of  $X$  and  $Y$ . If we let the concentrations of  $X$  and  $Y$  be denoted  $x = [X]$  and  $y = [Y]$ , and made some numerical simplifications, we imagined we could arrive at the following differential equation model:

$$\begin{aligned}\frac{dx}{dt} &= 1 - x \\ \frac{dy}{dt} &= 1 + x - 2y \\ x(0) &= 0, \quad y(0) = 1.\end{aligned}\tag{1}$$

Without dwelling too long on the physical motivation, we can image the positive terms as contributing to an increase in the amount of the associated variable, and the negative terms corresponding to a decrease in the corresponding amounts.

Although we did not classify it as such at the time, this is an example of a *first-order linear system of differential equations*, which is the final topic of the course. Before we explicitly state our objectives and methods for this topic area, there are a few things worth noting about this specific model:

- Even though there are two functions,  $x(t)$  and  $y(t)$ , we can check solutions in exactly the same way as before—by evaluating on the left-hand and right-hand sides of the equations. For this example, we can easily check that  $x(t) = 1 - e^{-t}$  and  $y(t) = 1 - e^{-t} + e^{-2t}$  is a solution because  $x(0) = 1 - e^{-(0)} = 0$ ,  $y(0) = 1 - e^{-(0)} + e^{-2(0)} = 1$ , and

$$\frac{dx}{dt} = \frac{d}{dt} [1 - e^{-t}] = e^{-t} = 1 - [1 - e^{-t}] = 1 - x(t)$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt} [1 - e^{-t} + e^{-2t}] \\ &= e^{-t} - 2e^{-2t} \\ &= 1 + [1 - e^{-t}] - 2[1 - e^{-t} + e^{-2t}] \\ &= 1 + x(t) - 2y(t).\end{aligned}$$

- Systems of differential equations can be classified in the same way as we classified single differential equations. This system is *first-order* because the highest derivative (of either  $x$  or  $y$ !) is first-order, and it is linear because all the terms we are solving for and their derivatives (i.e.  $x, y, x'$ , and  $y'$ ) appear isolated from one another. It is also nonhomogeneous because not every term involves an  $x$ , a  $y$ , or one of their derivatives (due to the additional constant “1”s which appear in both equations).
- We were able to solve this system at the time by noticing that the first equation depended on  $x$  only, and not  $y$ , and consequently we could solve for  $x(t)$  *before* consider  $y(t)$ . This is *not* a general property of systems of differential equations! In general, the variables are interdependent—that is to say, the equation for  $x'$  depends on  $x$  *and*  $y$ , and the equation for  $y'$  depends on  $x$  and  $y$  as well.

## 2 Conversion to First-Order Systems

Consider the general first-order system

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \dots, x_n) \\ \frac{dx_2}{dt} &= f_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, \dots, x_n).\end{aligned}\tag{2}$$

There are many nice properties of first-order systems but the main two are the following:

1. So long as the functions on the right-hand side are well-behaved (linear, homogeneous, etc.), there will be general techniques for obtaining a solution.

- The forward Euler and Runge-Kutta methods for numerical solutions can be immediately extended to first-order systems, so that numerically approximating solutions of differential equations in the form (2) is extremely straight-forward. (Note that this was not true for even second-order differential equations! Having only first-order derivatives is the key.)

The reasons we study *first-order* systems, however, goes far deeper than physical systems which can be directly written in the form (2). It turns out that almost every differential equations, regardless of order, can be written as a first-order system of differential equations with an appropriate introduction of variables. That is to say, in order to understand ordinary differential equations in general, it is (in almost all cases) sufficient to simply understand the form (2)!

This might be surprising, but the argument is actually straight-forward. It is also a procedure we will repeat throughout the remainder of this course (and likely in any subsequent courses involving ordinary differential equations!) so we will want to understand what is happening. We have the following steps:

- Consider a general  $n^{\text{th}}$  order differential equation written in the form

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)). \quad (3)$$

In other words, isolate the highest-order derivative.

- Make the variable substitutions  $x_1(t) = x(t)$ ,  $x_2(t) = x'(t)$ ,  $x_3(t) = x''(t)$ ,  $\dots$ ,  $x_n(t) = x^{(n-1)}(t)$ . In other words, assign a new variable to *everything except* the highest-order derivative.
- Notice that we have  $x'_1(t) = x_2(t)$ ,  $x'_2(t) = x_3(t)$ ,  $\dots$  by construction. In general, we have  $x'_i(t) = x_{i+1}(t)$  for  $i = 1, \dots, n - 1$ .
- Notice that this only works for the first  $n - 1$  variables. For the  $n^{\text{th}}$  equation must return to the original system. We notice that (3) implies  $x'_n(t) = f(t, x_1(t), x_2(t), \dots, x_n(t))$ .
- The *system* of *first-order* differential equations corresponding to (3) is

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= x_3 \\ &\vdots \\ x'_{n-1} &= x_n \\ x'_n &= f(t, x_1, x_2, \dots, x_n). \end{aligned} \quad (4)$$

- The initial conditions  $x(t_0) = b_1, x'(t_0) = b_2, \dots, x^{(n-1)}(t_0) = b_n$  become

$$x_1(t_0) = b_1, x_2(t_0) = b_2, \dots, x_n(t_0) = b_n.$$

**Example 1:** Rewrite the initial value problem

$$x''(t) + 4x'(t) + 4x(t) = \sin(2t) \quad (5)$$

$$x(0) = 5, x'(0) = -1 \quad (6)$$

as an initial value problem for a system of first-order differential equations.

**Solution:** We make the substitutions  $x_1(t) = x(t)$  and  $x_2(t) = x'(t)$ . This gives the relationship  $x'_1(t) = x_2(t)$ . In order to find an equation for  $x'_2(t)$  we need to rewrite (7) in the form

$$x''(t) = \sin(2t) - 4x'(t) - 4x(t).$$

We can see that this corresponds to the required form  $x''(t) = f(t, x(t), x'(t))$  which becomes  $x'_2(t) = f(t, x_1(t), x_2(t))$ . The desired system of first-order differential equations is therefore

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= 4x_1 - 4x_2 + \sin(2t) \end{aligned}$$

and the initial conditions  $x(0) = 5$  and  $x'(0) = -1$  become

$$x_1(0) = 5, x_2(0) = -1.$$

We have successfully transformed the original *second-order* differential equation into a system of *two* first-order differential equations!

**Example 2:** Rewrite the initial value problem

$$x'''(t) - x'(t)x(t) = 0 \quad (7)$$

$$x(0) = 1, x'(0) = 0, x''(0) = 0 \quad (8)$$

as an initial value problem for a system of first-order differential equations.

**Solution:** We follow the exact same procedure, but notice we have to assign *three* variables since this is a *third-order* differential equations. We set  $x_1(t) = x(t)$ ,  $x_2(t) = x'(t)$  and  $x_3(t) = x''(t)$ . It immediately follows

that  $x'_1(t) = x_2(t)$  and  $x'_2(t) = x_3(t)$ . It only remains to obtain a differential equation for  $x'_3(t) = x'''(t)$ . To obtain this, we rewrite the original differential equation as

$$x'''(t) = x'(t)x(t) = x_2(t)x_1(t).$$

It follows that our system of first-order differential equations is

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= x_1x_2\end{aligned}$$

with initial conditions

$$x_1(0) = 1, x_2(0) = 0, x_3(0) = 0.$$

While this process may not seem like much, it is the first step toward defining a general methodology for studying differential equations for very complicated systems! We should make a few quick notes:

- Notice that the order of the original system (3) corresponds to the number of variables in (4). This is a general property, which even generalized to *systems of higher-order differential equations*. For instance, if we have two differential equations, say a 3<sup>rd</sup> and a 5<sup>th</sup> order equation, respectively, then we can write this system as an equivalent *first-order* system in eight variables.
- Initial value problem for the system formulation (4) makes more intuitive sense than the original set-up (3). We could physically motivate why second-order systems required two initial conditions, but the mathematics was a little vague. Now it is clear! We have two initial conditions because there are two fundamental variables we need to specify.
- The first-order system formulation lends itself to very nice *geometric* tools. This is because the first-order derivative is very easy to interpret graphically—it is the slope of the solution function at the given point. We just need to draw a line pointed in an appropriate direction. So, if we formulate higher-order differential equations as systems of first-order differential equations, we will be able to (easily!) draw pictures. (This is very strong analogue with the first-order systems at the start of the course, where we always tried to reconcile our algebraic results with some sketch.)

### 3 First-Order Linear Homogeneous Systems of Differential Equations in Two Variables

We start by considering systems of two linear, first-order homogeneous differential equations with constant coefficients:

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy.\end{aligned}\tag{9}$$

Before considering how we might *find* an analytic solution  $(x(t), y(t))$  to such a system, let's first ask a more basic question: What *can* a system like this do? Let's consider this question for a geometrical point of view; in other words, let's try to draw a *picture*. We make the following observations:

- The system is first-order so that, at every point  $(x_0, y_0)$  in the  $(x, y)$ -plane we know whether the solution through the point  $(x_0, y_0)$  is pointed right or left ( $x'(t) > 0$  or  $x'(t) < 0$ ) or up or down ( $y'(t) > 0$  or  $y'(t) < 0$ ).
- We know the equation  $x'(t) = 0$  corresponds to  $ax + by = 0$  or  $y = -(a/b)x$  and  $y'(t) = 0$  corresponds to  $y = -(c/d)x$ . In other words, we know exactly where the solution  $(x(t), y(t))$  is completely flat ( $y'(t) = 0$ ) or completely vertical ( $x'(t) = 0$ ).

**Example 1:** Consider the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= -x + 3y \\ \frac{dy}{dt} &= 3x - y.\end{aligned}$$

We can easily determine that

$$\frac{dx}{dt} = 0 \implies y = \frac{1}{3}x$$

and

$$\frac{dy}{dt} = 0 \implies y = 3x.$$

The question then becomes what happens in the regions between these two lines. It should not take too much convincing that, if we only consider arrows pointing in the dominant directions (NW, NE, SW, SE, say) that we

arrive at the picture given by Figure 1(a).

**Example 2:** Consider the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= -x + 5y \\ \frac{dy}{dt} &= -2x + y.\end{aligned}$$

We can easily determine that  $x' = 0$  implies  $y = (1/5)x$ , and that  $y' = 0$  implies  $y = 2x$ . When we consider the orthants, we end up with a picture that looks something like Figure 1(b).

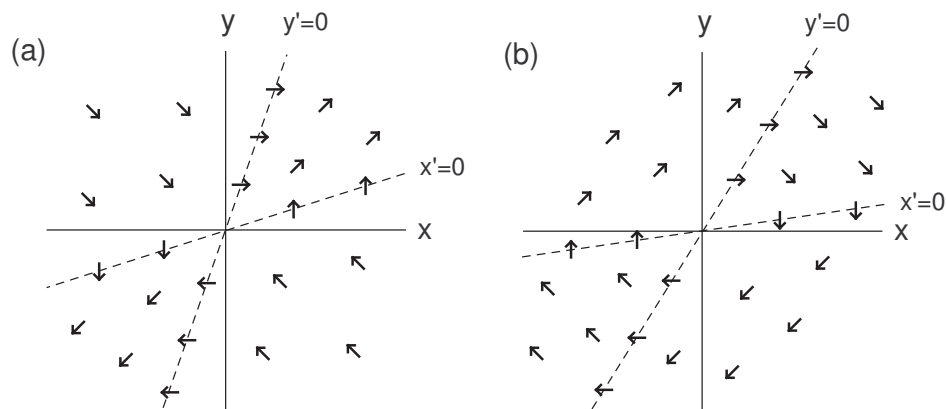


Figure 1: A rough sketch of the two example systems. Even without solving the equations, we can get some sense about how the solutions behave!

Without even attempting to solve the system of differential equations, we can tell very important things about the types of behaviors we might encounter. It looks like the solutions of the first system originate somewhere in the top-left or bottom-right, pool together, then travel toward either the top-right or the bottom-left. Solutions of the second system, by contrast, appear to spiral around  $(0,0)$ , although it is unclear whether they approach  $(0,0)$  or drift away.

## 4 General solutions to Linear Systems in Two Variables

In order to investigate how we might find a solution to these systems, or a general system such as (9), let's make an observation about the form of the equation. Notice that the right-hand side can be written in a condensed form by appealing to matrix multiplication. In particular, we can write the system (9) as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \quad (10)$$

where

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{and } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This form suggests immediately that results from linear algebra will be relevant for solving *systems* of differential equations. In fact, this intuition is completely justified, and we will see that we already have all of the tools needed to completely solve systems of DEs of the form (10)!

Before we get there, however, let's try to build some intuition. The first order equation (10) is a vector/matrix equation, but it looks eerily similar to the first order equation

$$\frac{dx}{dt} = ax$$

which we know has solution  $x(t) = Ce^{at}$ . The question then becomes, can we extend our standard algebra result by substituting matrix algebra instead? What are the terms going to be? Can we write  $e^{At}$  for a matrix  $A$ ? (We can, but won't attempt to do this, at least not yet.) Is there some other way we can extend the solution  $x(t)$  to the vector solution  $\mathbf{x}(t)$ ?

Consider the following set-up. We guess a solution  $\mathbf{x}(t)$  with the general exponential form  $e^{at}$ , but we allow the components of  $\mathbf{x}(t)$  to vary according to some constant vector. In other words, we write  $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$  for some  $\lambda \in \mathbb{R}$ . This keeps the general intuition that the solution is exponential while allowing that each equation may be slightly different.

Now let's check the matrix equation (10)! We have

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}[\mathbf{v}e^{\lambda t}] = [\lambda\mathbf{v}]e^{\lambda t}$$

and

$$A\mathbf{x} = A[\mathbf{v}e^{\lambda t}] = [A\mathbf{v}]e^{\lambda t}.$$



It follows that we need

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \implies [\lambda\mathbf{v}]e^{\lambda t} = [A\mathbf{v}]e^{\lambda t}.$$

After dividing by  $e^{\lambda t}$  (which is never zero) and rearranging, we have

$$A\mathbf{v} = \lambda\mathbf{v}.$$

If you get the sense that we have seen this equation before, it is because we have. This is exactly the eigenvalue/eigenvector equation for the matrix  $A$ . In this context of differential equations, this tells us that eigenvalues  $\lambda_{1,2}$  and corresponding eigenvectors  $\mathbf{v}_{1,2}$  give us solutions to (10) of the form  $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$  and  $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$ . With a little bit of work, we should be able to convince ourselves that any solution of the form

$$\mathbf{x}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t}$$

will also satisfy the differential equation. In other words, we may construct the *general* solution out of *individual* solutions! This is the same trick we employed for second-order linear differential equations.

**Example 1:** Find the general solution to the first order system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -x + 3y \\ \frac{dy}{dt} &= 3x - y. \end{aligned}$$

**Solution:** Notice that we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{with } A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

We can quickly compute that the eigenvalues are given by  $(-1 - \lambda)(-1 - \lambda) - 9 = \lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2) = 0$  so that  $\lambda_1 = -4$  and  $\lambda_2 = 2$ . The corresponding eigenvectors are  $\mathbf{v}_1 = (1, -1)$  and  $\mathbf{v}_2 = (1, 1)$ . It follows that we have two solutions of the form  $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$  and  $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$ . It follows that the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Remarkably, knowing how to compute eigenvalues and eigenvectors completely solves the problem! We also get to complete our earlier picture. Since  $C_1(1, -1)e^{-4t} \rightarrow 0$  as  $t \rightarrow \infty$ , we have that the solution gets closer to  $C_2(1, 1)e^{2t}$  as time goes on. In other words, solution approach the direction of the vector  $(1, 1)$ . We can quickly confirm that this is where the arrows pointed in the picture in Figure 1(a).

Of course, things are not always quite so easy. There are three basic cases for eigenvalues: distinct real eigenvalues, repeated eigenvalues, and complex eigenvalues. The solution forms for the latter two cases differ from what we have just obtained. But we already have the tools for resolving these cases! If we have a repeated eigenvalue, then we do not get a full set of eigenvectors and so do not get a full set of solutions. But we have already seen how to generate further independent solutions—we just add a factor of  $t$ ! Similarly, if we encounter complex eigenvalues  $\lambda = \alpha \pm \beta i$ , we must take a particular combination of *complex-valued* solutions to get *real-valued* solutions in terms of  $e^{\alpha t} \sin(\beta t)$  and  $e^{\alpha t} \cos(\beta t)$ .

In any case, the solution for the  $2 \times 2$  differential equation cases can be completely determined by the eigenvalues and eigenvectors in the following way:

1. **Two real distinct eigenvalues (or a repeated eigenvalue with two distinct eigenvectors)** - If we have two distinct eigenvalues  $\lambda_1$  and  $\lambda_2$  corresponding to  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively, the solution to (10) is given by

$$\mathbf{x}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t}.$$

Similarly, if there is a repeated eigenvalue ( $\lambda = \lambda_1 = \lambda_2$ ) but two linearly independent eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , we have

$$\mathbf{x}(t) = e^{\lambda t} (C_1 \mathbf{v}_1 + C_2 \mathbf{v}_2).$$

2. **Repeated eigenvalue, one eigenvector** - If we have a repeated eigenvalue  $\lambda = \lambda_1 = \lambda_2$  but only one eigenvector  $\mathbf{v} \in \mathbb{R}^2$ , we have the general solution

$$\mathbf{x}(t) = (C_1 \mathbf{v} + C_2 (t\mathbf{v} + \mathbf{w})) e^{\lambda t}$$

where  $\mathbf{w} \in \mathbb{R}^2$  is a **generalized eigenvector** satisfying

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

3. **Complex eigenvalues** - If we have a complex eigenvalue  $\lambda = \alpha + i\beta$  corresponding to a complex eigenvector  $\mathbf{v} = \mathbf{a} + i\mathbf{b}$  then the general solution is given by

$$\begin{aligned}\mathbf{x}(t) &= C_1 e^{\alpha t} (\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)) \\ &\quad + C_2 e^{\alpha t} (\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)).\end{aligned}$$

There are a few notes worth making about these solutions:

1. It is clear that exponentials factor very heavily in the solutions of linear systems of differential equations. We also notice that, in terms of limiting behavior, these exponentials dominate the behavior (i.e. they asymptotically overwhelm the factor  $t$  in case (2), and the trigonometric functions in (3)). That is to say, the long-term behavior is determined by the exponentials, so that trajectories tend to decay (i.e. approach  $(0,0)$ ) if  $\text{Re}(\lambda) < 0$  and blow up (i.e. go away from  $(0,0)$ ) if  $\text{Re}(\lambda) > 0$ .
2. The case when  $\lambda = 0$  is somewhat special, but it is worth noting that the solution forms for case (1) and (2) still hold, but that the exponential becomes a constant.
3. What is implicit in this result, but has not been stated explicitly, is that *all* solutions can be represented in one of these forms. That is to say, every solution can be written in the form

$$\mathbf{x}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t)$$

for some fundamental solution solutions  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$ , where the form of these solutions are as above (Theorem 7.4.2 in text). It also follows that, for any initial conditions  $x(0) = x_0$  and  $y(0) = y_0$  we can solve uniquely for  $C_1$  and  $C_2$  so the solutions to initial value problems are *unique* (Theorem 7.1.2 in the text). The parallel with second-order differential equations should be striking, but not surprising, since we know second-order differential equations can be written as a system of two first-order differential equations.

**Example 1:** Determine the solution of

$$\begin{aligned}\frac{dx}{dt} &= -x + 3y, & x(0) &= 1 \\ \frac{dy}{dt} &= 3x - y, & y(0) &= 1.\end{aligned}$$

We have already determined that the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

It remains to use the initial conditions to solve for  $C_1$  and  $C_2$ . We have that  $x(0) = 1$  and  $y(0) = 1$  so that at  $t = 0$  we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can rewrite this as

$$\begin{aligned} -C_1 + C_2 &= 1 \\ C_1 + C_2 &= 1. \end{aligned}$$

We can solve this by a number of methods to determine that  $C_1 = 0$  and  $C_2 = 1$  so that the particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

**Example 2:** Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= -x + 5y, & x(0) &= 1 \\ \frac{dy}{dt} &= -2x + y, & y(0) &= 1. \end{aligned}$$

To find the eigenvalues, we realize

$$A = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} -1 - \lambda & 5 \\ -2 & 1 - \lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(-1 - \lambda)(1 - \lambda) + 10 = \lambda^2 + 9 = 0.$$

It follows that  $\lambda = \pm 3i$ . We need to find the eigenvectors corresponding to these values. We have

$$(A - (3i)I) = \begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i \end{bmatrix}.$$

To find the corresponding eigenvector, we row reduce to get

$$\begin{aligned} \left[ \begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array} \right] &\xrightarrow{(-1+3i)R_1} \left[ \begin{array}{cc|c} (-1-3i)(-1+3i) & 5(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{cc|c} 10 & -5+15i & 0 \\ -2 & 1-3i & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & -\frac{1}{2} + \frac{3}{2}i & 0 \\ 0 & 0 & 0 \end{array} \right] \end{aligned}$$

so that  $\mathbf{v} = (1 - 3i, 2)$ . We rewrite this as

$$\mathbf{v} = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

We set  $\alpha = \operatorname{Re}(\lambda) = 0$  and  $\beta = \operatorname{Im}(\lambda) = 3$  and  $\mathbf{a} = \operatorname{Re}(\mathbf{v}) = (1, 2)$  and  $\mathbf{b} = \operatorname{Im}(\mathbf{v}) = (-3, 0)$ . It follows that the general solution is

$$\begin{aligned} \mathbf{x}(t) &= C_1 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &\quad + C_2 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right). \end{aligned}$$

To solve for  $C_1$  and  $C_2$ , we utilize the initial conditions  $x(0) = 1$  and  $y(0) = 1$ . At  $t = 0$  we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

so that we have

$$\begin{aligned} C_1 - 3C_2 &= 1 \\ 2C_1 &= 1 \end{aligned}$$

It follows immediately that  $C_1 = 1/2$  and  $C_2 = -1/6$  so we have

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &\quad - \frac{1}{6} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(3t) + \frac{1}{3} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \sin(3t) \end{aligned}$$

**Example 3:** Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= x - 4y, & x(0) &= -1 \\ \frac{dy}{dt} &= x - 3y, & y(0) &= 2. \end{aligned}$$

To find the eigenvalues, we realize

$$A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} 1 - \lambda & -4 \\ 1 & -3 - \lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(1 - \lambda)(-3 - \lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

so that  $\lambda = -1$  is a repeated eigenvalue. To check for the eigenvector(s) corresponding to this value, we have

$$(A - (-1)I) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

To find the corresponding eigenvector, we row reduce to get

$$\left[ \begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

so that  $\mathbf{v} = (2, 1)$ . We notice that we have not obtained eigenvectors, so that we need to look for a generalized eigenvector  $\mathbf{w}$ . We have

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \implies \left[ \begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array} \right].$$

If we set  $w_2 = t$ , we see that  $w_1 = 1 + 2t$  so that we have

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Setting  $t = 0$ , we have  $\mathbf{w} = (1, 0)$ .

It follows that the general solution is given by

$$\mathbf{x}(t) = \left( C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left( t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t}$$

To solve for  $C_1$  and  $C_2$ , we utilize the initial conditions  $x(0) = -1$  and  $y(0) = 2$ . At  $t = 0$  we have

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which implies

$$\begin{aligned} 2C_1 + C_2 &= -1 \\ C_1 &= 2. \end{aligned}$$

It follows that  $C_1 = 2$  and  $C_2 = -5$ . It follows that the solution is

$$\begin{aligned}\mathbf{x}(t) &= \left( 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5 \left( t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t} \\ &= \left( \begin{bmatrix} -1 \\ 2 \end{bmatrix} - t \begin{bmatrix} 10 \\ 5 \end{bmatrix} \right) e^{-t}\end{aligned}$$

## 5 Phase Portrait

Now that we have a sense of what the solutions look like, we can construct a much more detailed picture. In fact, we can completely enumerate the possible qualitatively different cases we found when we considered the analytic solutions. We can break things apart something like this (for representative pictures, see Figure 2):

1. **Two real distinct eigenvalues (or repeated eigenvalues with two distinct eigenvectors)**
  - (a) If both eigenvalues are positive ( $\lambda_1 > 0$  and  $\lambda_2 > 0$ ) we say  $(0, 0)$  is an *unstable node* or *source*.
  - (b) If both eigenvalues are negative ( $\lambda_1 < 0$  and  $\lambda_2 < 0$ ) we say  $(0, 0)$  is a *stable node* or *sink*.
  - (c) If the eigenvalues have opposite sign, we say  $(0, 0)$  is a *saddle point*.
2. **Repeated eigenvalue, one eigenvector**
  - (a) If the repeated eigenvalue is positive ( $\lambda > 0$ ) we say  $(0, 0)$  is a *degenerate source*.
  - (b) If the repeated eigenvalue is negative ( $\lambda < 0$ ) we say  $(0, 0)$  is a *degenerate sink*.
3. **Complex eigenvalues**
  - (a) If the real part of the eigenvalue is positive ( $\alpha > 0$ ) we say  $(0, 0)$  is an *unstable spiral* or *source spiral*.
  - (b) If the real part of the eigenvalue is negative ( $\alpha < 0$ ) we say  $(0, 0)$  is a *stable spiral* or *sink spiral*.
  - (c) If the real part of the eigenvalue is zero ( $\alpha = 0$ ) we say  $(0, 0)$  is a *center*.

#### 4. Zero eigenvalue

- (a) If there is a zero eigenvalue, we say that the system is *degenerate* (there is a line of fixed points through  $(0, 0)$ ).

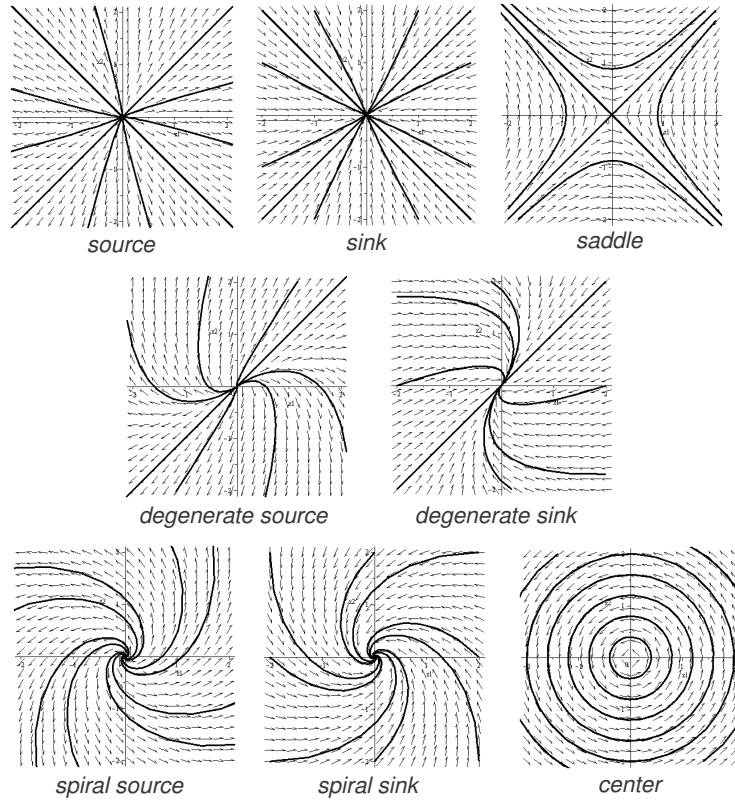


Figure 2: Canonical pictures for the various cases of two-dimensional linear autonomous differential equations.