MATH 319, WEEK 15:
The Fundamental Matrix,
Non-Homogeneous Systems of Differential Equations

1 Fundamental Matrices

Consider the problem of determining the particular solution for an *ensemble* of initial conditions. For instance, suppose we are considering the differential equation

$$\frac{dx}{dt} = -x + 3y$$
$$\frac{dy}{dt} = 3x - y$$

which we know has the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Suppose, however, that instead of determining the particular solution for a single initial condition $x(0) = x_0$, $y(0) = y_0$, we wish to determine an ensemble of particular solutions for a variety conditions. As things stand now, we are required to solve for C_1 and C_2 individually *each time* we wish to apply an initial condition. This could be a lot of work!

Fortunately, this will not be required, but we will have to make use of a little linear algebra in order to see how to get around the problem. First of all, we should recognize that we can write the solution in the equivalent form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

In other words, we can write the equation in the matrix form

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \tag{1}$$

where $\Psi(t)$ is the 2×2 matrix with the fundamental solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ along the columns.

Our goal is to relate the undetermined coefficients $\mathbf{c} \in \mathbb{R}^2$ to the initial conditions $\mathbf{x}(0)$ in general. This equation tells us exactly how to do that! We have

$$\mathbf{x}(0) = \Psi(0)\mathbf{c} \quad \Longrightarrow \quad \mathbf{c} = \Psi(0)^{-1}\mathbf{x}(0) = \Psi^{-1}(0)\mathbf{x}_0 \tag{2}$$

where $\Psi(0) \in \mathbb{R}^{2\times 2}$ is the matrix $\Psi(t)$ evaluated at zero and $\Psi^{-1}(0)$ is the inverse of this matrix. We can now plug (2) into (1) to get

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 \tag{3}$$

where $\Phi(t) = \Psi(t)\Psi^{-1}(0) \in \mathbb{R}^{2\times 2}$. This is exactly what was wanted! Once we have determined the matrix $\Phi(t)$, we have a relationship which immediately gives the solution $\mathbf{x}(t)$ for an arbitrary initial vector $\mathbf{x}_0$.

For the example, we have that

$$\Psi(t) = \begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix} \quad \Longrightarrow \quad \Psi(0) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that

$$\Psi^{-1}(0) = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It follows that

$$\begin{aligned} \Phi(t) = \Psi(t)\Psi^{-1}(0) &= \frac{1}{2}\begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{-4t} + e^{2t} & -e^{-4t} + e^{2t} \\ -e^{-4t} + e^{2t} & e^{-4t} + e^{2t} \end{bmatrix} \end{aligned}$$

and therefore the solution can be written in the form $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-4t} + e^{2t} & -e^{-4t} + e^{2t} \\ -e^{-4t} + e^{2t} & e^{-4t} + e^{2t} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

As expected, the result relates the solutions directly to the initial conditions. We no longer have to compute the constants C_1 and C_2 at every iteration!

We stop to make the following notes:

1. The matrices $\Psi(t)$ and $\Phi(t)$ are called *fundamental matrices*. The matrix $\Phi(t)$ will be of particular concern to us as it has many exceptionally useful properties. (More on this in the next section.)

2. It follows from (3) that the fundamental matrix $\Phi(t)$ has the property $\Phi(0) = I$. (Otherwise, we would have $\mathbf{x}(0) \neq \Phi(0)\mathbf{x}_0$.) This can be verified in the above example since we have

$$\Phi(0) = \frac{1}{2}\left[\begin{array}{cc} e^{-4(0)} + e^{2(0)} & -e^{-4(0)} + e^{2(0)} \\ -e^{-4(0)} + e^{2(0)} & e^{-4(0)} + e^{2(0)} \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

3. The matrix $\Phi(t)$ also have the property that it satisfies the differential equation $\Phi'(t) = A\Phi(t)$, since each column of $\Phi(t)$ is a solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Example 2: Determine the fundamental matrix $\Phi(t) \in \mathbb{R}^{2\times 2}$ for the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - 4y \\ \frac{dy}{dt} &= x - 3y. \end{aligned}$$

Solution: We determined last week that this system of differential equations has the general solution

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left(C_1 \left[\begin{array}{c} 2 \\ 1 \end{array}\right] + C_2 \left(t \left[\begin{array}{c} 2 \\ 1 \end{array}\right] + \left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right)\right) e^{-t}.$$

We now want to determine the fundamental matrix $\Phi(t) = \Psi(t)\Psi^{-1}(0)$ where $\Psi(t)$ is the matrix with the fundamental solutions above along the columns. We will have to be a little careful when establishing our fundamental solutions—they are component-wise coefficients of the undetermined constants C_1 and C_2. In this case, we have that

$$\Psi(t) = \left[\begin{array}{cc} 2e^{-t} & e^{-t} + 2te^{-t} \\ e^{-t} & te^{-t} \end{array}\right].$$

It follows that

$$\Psi(0) = \left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array}\right] \quad \Longrightarrow \quad \Psi^{-1}(0) = \left[\begin{array}{cc} 0 & 1 \\ 1 & -2 \end{array}\right].$$

Consequently, we have

$$\begin{aligned} \Phi(t) = \Psi(t)\Psi^{-1}(0) &= \left[\begin{array}{cc} 2e^{-t} & e^{-t} + 2te^{-t} \\ e^{-t} & te^{-t} \end{array}\right] \left[\begin{array}{cc} 0 & 1 \\ 1 & -2 \end{array}\right] \\ &= \left[\begin{array}{cc} e^{-t} + 2te^{-t} & -4te^{-t} \\ te^{-t} & e^{-t} - 2te^{-t} \end{array}\right] \end{aligned}$$

2 Matrix Exponentials

We will now take a step backward to take a step forward. Reconsider the general linear and homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t). \tag{4}$$

If we carry through with our earlier analogy with a single first order equation of the form $x'(t) = ax(t)$, we would recognize the general form of the solution as $x(t) = x_0 e^{at}$. So perhaps there is some sense in which we could write

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

as the solution for our system (4). (Note that this is very close to the solution $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ we just discovered!)

The question then becomes: how might we define e^{At}? It should be clear that we cannot simply take the exponential of each term in the matrix as this does not generally lead a function $\mathbf{x}(t)$ which is a solution of (4). Consider the following two possible approaches:

1. Recall that the exponential e^{at} can be expanded as the Taylor series

$$e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

This suggests that, instead of defining the matrix exponent e^{At} directly, we define it as the infinite series

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \tag{5}$$

(Notice here that $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc.) We should have some healthy skepticism about this formula. After all, it is an infinite sum, which we have no hope of computing explicitly in our lifetimes. The best we might hope for is some result regarding convergence. But before we give up hope on this interpretation entirely, consider the following observation: It is a solution of the system of differential equation! To check this, we can simply evaluate. We clearly have

$$\mathbf{x}(0) = e^{A(0)}\mathbf{x}_0 = (I + A(0) + \frac{1}{2!}A^2(0)^2 + \cdots)\mathbf{x}_0 = \mathbf{x}_0.$$

On the left-hand side of the system, we have

$$\begin{aligned} \frac{d}{dt}\left(e^{At}\mathbf{x}_0\right) &= \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)\mathbf{x}_0 \\ &= \left(A + A^2t + \frac{1}{2!}A^3t^2 + \cdots\right)\mathbf{x}_0 \end{aligned}$$

and on the right-hand side we have

$$\begin{aligned} A\left(e^{At}\mathbf{x}_0\right) &= A\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)\mathbf{x}_0 \\ &= \left(A + A^2t + \frac{1}{2!}A^3t^2 + \cdots\right)\mathbf{x}_0. \end{aligned}$$

However uncomfortable this definition may seen, we cannot escape the implication that it is meaningful! (Whether it is *useful* is another discussion entirely.)

2. Recall our earlier discussion, where we defined the fundamental matrix $\Phi(t) = \Psi(t)\Psi^{-1}(0)$ and noted that the function $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ was a solution of our differential equation and that it satisfies the initial conditions (since $\Phi(0) = I$). In other words, we really have *two* functions which we are claiming are solutions of the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{6}$$

specifically, $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ and $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$. But how can this be? How can an initial value problem has two solutions? The answer is that it *can't*. It is a well-known fact (Theorem 7.1.2 of text) that the initial value problem (6) always has a *unique* solution. We are inescapably drawn to conclude that

$$e^{At} = \Phi(t). \tag{7}$$

This equation should be surprising! What it will allow us to do is interpret the fundamental matrix $\Phi(t)$ in two ways: a solution of a linear system of differential equations (right-hand side of (7)), or as a matrix exponential with many of the properties and intuitions which come from the standard exponential function (left-hand side of (7)).

Example: Determine the matrix exponential e^{At} for the matrix

$$A = \left[\begin{array}{cc} -1 & 5 \\ -2 & 1 \end{array}\right].$$

Solution: Without the preceding motivation, we might be a little lost. By definition, we have

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

which, beyond a few terms, we have no hope of computing explicitly beyond a few terms in the series. Nevertheless, this *is* the definition—we are now allowed to take the exponential of A component-wise as this will produce an incorrect result.

What we realize given the previous discussion is that $e^{At} = \Phi(t)$ where $\Phi(t)$ is the particular fundamental solution of the differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

with the property $\Phi(0) = I$. In other words, all we need to do is find $\Phi(t)$ for the first-order system

$$\begin{aligned} \frac{dx}{dt} &= -x + 5y \\ \frac{dy}{dt} &= -2x + y. \end{aligned}$$

We found last week that this system had the general solution

$$\begin{aligned} \mathbf{x}(t) = C_1 &\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &+ C_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right). \end{aligned}$$

The fundamental matrix $\Psi(t)$ is therefore given by

$$\Psi(t) = \begin{bmatrix} \cos(3t) + 3\sin(3t) & -3\cos(3t) + \sin(3t) \\ 2\cos(3t) & 2\sin(3t) \end{bmatrix}$$

so that

$$\Psi(0) = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} e^{At} = \Phi(t) &= \frac{1}{6}\begin{bmatrix} \cos(3t) + 3\sin(3t) & -3\cos(3t) + \sin(3t) \\ 2\cos(3t) & 2\sin(3t) \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 6\cos(3t) - 2\sin(3t) & 10\sin(3t) \\ -4\sin(3t) & 6\cos(3t) + 2\sin(3t) \end{bmatrix} \\ &= \begin{bmatrix} \cos(3t) - \frac{1}{3}\sin(3t) & \frac{5}{3}\sin(3t) \\ -\frac{2}{3}\sin(3t) & \cos(3t) + \frac{1}{3}\sin(3t) \end{bmatrix} \end{aligned}$$

Notice that, as expected, we have $e^{A(0)} = \Phi(0) = I$. We have accomplished our task, without having to ever directly consider properties of the matrix exponential e^{At}!

3 Non-Homogeneous Linear Systems of Differential Equations with Constant Coefficients

Now consider being asked to solve the differential equation

$$\begin{aligned} \frac{dx}{dt} &= 1 - x \\ \frac{dy}{dt} &= 1 + x - 2y \end{aligned}$$

by using the matrix algebra methods we have been employing over the past two weeks.

We would notice quickly that we could rewrite the left-hand side as a vector derivative $\mathbf{x}'(t) = (x'(t), y'(t))$ and collect most of the right-hand side as

$$A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

But what do we do with the extra terms in the equation (i.e. the "1"s)? They do not fit into either of these forms! The answer is that the best we can do is writen them as a separate vector, so that we have

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In general, we may consider differential equations of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t). \tag{8}$$

Such systems are called *non-homogeneous* for the same reason as we called second-order system homogeneous or non-homogeneous—there are terms which affect the system which are independent of solutions and their derivatives (i.e. x', y', x, and y). How might we go about solving such a system?

Let's consider one of the first integration techniques we learned for first-order differential equations. If we had a general first-order linear differential equation

$$\frac{dx}{dt} = ax(t) + g(t)$$

we would re-arrange the expression as

$$\frac{dx}{dt} - ax(t) = g(t)$$

and then obtain the integrating factor

$$\mu(t) = e^{\int -a \; dt} = e^{-at}$$

so that we could write

$$\frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}g(t).$$

Depending on whether we were interested in the general solution or a particular solution (i.e. if we have initial conditions), we could either compute the indefinite integral to get

$$x(t) = Ce^{at} + e^{at}\int e^{-as}g(s)\; ds$$

or integrate explicitly from $s = t_0$ to $s = t$ to obtain

$$x(t) = x_0e^{at} + e^{at}\int_{t_0}^{t} e^{-as}g(s)ds$$

where $x(t_0) = x_0$.

There is nothing stopping us from attempting this technique with our linear system (9)! We may write the expression as

$$\frac{d\mathbf{x}}{dt} - A\mathbf{x}(t) = \mathbf{g}(t)$$

and then obtain the integrating factor $\mu(t) = e^{-At}$. (We will suspend discussion of the matrix exponential until later, just pausing now to acknowledge that we have seen this object before and know how to handle it!) It follows that we have

$$\frac{d}{dt}\left[e^{-At}\mathbf{x}(t)\right] = e^{-At}\mathbf{g}(t).$$

If we are interested in the general solution, we can compute the indefinite integral and rearrange to get

$$\mathbf{x}(t) = e^{At}\mathbf{c} + e^{At}\int e^{-As}\mathbf{g}(s)\; ds. \tag{9}$$

If the initial conditions are specified, we can integrate from $s = t_0$ to $s = t$ to get

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_{t_0}^{t} e^{-As}\mathbf{g}(s)\, ds. \tag{10}$$

If we had encountered this "solution" before our previous discussion of matrix exponentials, we would be very distressed! We would like to meaningfully define the matrix exponential e^{At} in some closed form, but would only have the infinite series definition available to us. We now know, however, that $e^{At} = \Phi(t)$ where $\Phi(t)$ is the fundamental matrix of the corresponding *homogeneous* system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

which satisfies the condition $\Phi(0) = I$. We can rewrite (9) and (10) to take advantage of this simplification. Allowing the property $e^{-At} = [e^{At}]^{-1} = \Phi^{-1}(t)$ (true, but beyond the scope of this course!), for problems without initial conditions we have the formulas

$$\begin{aligned} & \mathbf{x}(t) = \Phi(t)\mathbf{c} + \Phi(t)\int \Phi^{-1}(s)\mathbf{g}(s)\, ds \\ \Longrightarrow\quad & \mathbf{x}(t) = \Psi(t)\tilde{\mathbf{c}} + \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\, ds \end{aligned} \tag{11}$$

where $\mathbf{c} = (C_1, C_2)$ and $\tilde{\mathbf{c}} = (\tilde{C}_1, \tilde{C}_2)$ are vectors of undetermined constants, while for problems with initial conditions we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{g}(s)\, ds \tag{12}$$

In other words, now that we understand the matrix exponential e^{At}, we can solve first-order linear *systems* of differential equations by exactly the same method integrating factor we solved first-order linear differential equations with during the second week of the course!

We stop now to make a few notes:

1. The two equations in (11) are equivalent (details below) but it is generally easier to use the $\Psi(t)$ derived from the standard solution from than the modified $\Phi(t)$. That is to say, the second formula is typically *easier to solve*.

2. It will be very important to remember when to integrate the *definite* integral from $s = t_0$ (usually $t_0 = 0$) to $s = t$ when initial conditions are specified! Otherwise the formula will fail.

3. (Technical details ahead!) The equivalence of the two formulas in (11) and (12) can be seen by noting that $\Phi(t) = \Psi(t)\Psi^{-1}(0)$ and $\Phi^{-1}(t) = [\Psi(t)\Psi^{-1}(0)]^{-1} = \Psi(0)\Psi^{-1}(t)$ (properties of matrices, but beyond the scope of the course). We consequently have

$$\begin{aligned}\Phi(t)\int \Phi^{-1}(s)\mathbf{g}(s) &= \Psi(t)\Psi^{-1}(0)\int \Psi(0)\Psi^{-1}(s)\mathbf{g}(s)\,ds \\ &= \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\,ds\end{aligned}$$

since $\Psi^{-1}(0)\Psi(0) = I$. We have the other equivalence of $\Phi(t)\mathbf{c} = \Psi(t)\Psi^{-1}(0)\mathbf{c} = \Psi(t)\tilde{\mathbf{c}}$ where $\tilde{\mathbf{c}} = \Psi^{-1}(0)\mathbf{c}$. The key thing to note is that, since the vector of constants $\mathbf{c}$ is arbitrary, $\tilde{\mathbf{c}}$ is just as arbitrary.

Example: Let's see how this applies to our previous example. Since we have initial conditions, we will want to find the fundamental matrix $\Phi(t)$. We therefore solve the linear system of differential equations $\mathbf{x}(t) = A\mathbf{x}(t)$ with the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}.$$

We can quickly compute that the characteristic polynomial is $(-1-\lambda)(-2-\lambda) = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues. For $\lambda = -1$ we have the eigenvector $\mathbf{v}_1 = (1,1)$ and for $\lambda = -2$ we have the eigenvector $\mathbf{v}_2 = (0,1)$. It follows that the general solution is

$$\mathbf{x}(t) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-2t}.$$

This gives the fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{bmatrix}.$$

In order to compute $\Phi(t)$, we evaluate

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad\Longrightarrow\quad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

so that

$$\begin{aligned}\Phi(t) = \Psi(t)\Psi^{-1}(0) &= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix}\end{aligned}$$

To find the general solution $\mathbf{x}(t)$ by the formula (10) we need to compute the matrix $\Phi^{-1}(t)$. First of all, we have

$$\det(\Phi(t)) = e^{-t}e^{-2t} - (e^{-t} - e^{-2t})(0) = e^{-3t}.$$

It follows that

$$\Phi^{-1}(t) = \frac{1}{e^{-3t}}\begin{bmatrix} e^{-2t} & 0 \\ -e^{-t} + e^{-2t} & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{2t} & e^{2t} \end{bmatrix}.$$

We now evaluate the integral. Since $t_0 = 0$, we have

$$\begin{aligned} \int_0^t \Phi^{-1}(s)\mathbf{g}(s)\, ds &= \int_0^t \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} e^{s} \\ e^{s} \end{bmatrix} ds \\ &= \begin{bmatrix} e^{t} - 1 \\ e^{t} - 1 \end{bmatrix} \end{aligned}$$

We can now compute

$$\begin{aligned} \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{g}(s)\, ds &= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} e^{t} - 1 \\ e^{t} - 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - e^{-t} \\ 1 - e^{-t} \end{bmatrix} \end{aligned}$$

It follows that the general solution is

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{g}(s)\, ds \\ &= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 1 - e^{-t} \\ 1 - e^{-t} \end{bmatrix}. \end{aligned}$$

In particular, if we use the initial conditions $x(0) = 0$ and $y(0) = 1$ then we obtain the solution

$$\begin{aligned} x(t) &= 1 - e^{-t} \\ y(t) &= 1 - e^{-t} + e^{-2t} \end{aligned}$$

as we had before. Despite the amount of work we have done, we should be encouraged that none of the individual steps were particularly challenging, and that the result obtained is consistent with our earlier work!

Note: One thing worth noting is that we computed

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix} \quad \text{and} \quad \Phi^{-1}(t) = \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{2t} & e^{2t} \end{bmatrix}.$$

This should be striking—the second matrix is simply the first with the argument $-t$ instead of t. That is to say, we have $\Phi^{-1}(t) = \Phi(-t)$. We might wonder if this is the general case. The answer in general is a definitive *NO*. For a general fundamental matrix $\Psi(t)$ it is not true that $\Psi^{-1}(t) = \Psi(-t)$. That said, it is true in the very specific cases we will be consider—that of the fundamental matrix $\Phi(t)$. This is due, in analogy, to the identity $\Phi(t) = e^{At}$ and the property that $\Phi(-t) = e^{-At} = [e^{At}]^{-1} = \Phi(t)^{-1}$. In other words, it is because we can relate $\Phi(t)$ to a matrix exponential that we are justified in believing $\Phi(t)^{-1} = \Phi(-t)$.

Example 2: Find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -4x - 3y + e^{-t} \\ \frac{dy}{dt} &= 3x + 2y + e^{-t}. \end{aligned}$$

Solution: Since we are looking for a general solution, any fundamental solution $\Psi(t)$ will work. We first solve the homogeneous equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ where

$$A = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}.$$

We have the characteristic polynomial $(-4-\lambda)(2-\lambda)+9 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$. It follows that $\lambda = -1$. So we have a repeated eigenvalue. We compute the corresponding eigenvector quickly by solving $(A+I)\mathbf{v} = \mathbf{0}$. This simplifies to

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $v_1 + v_2 = 0$. Setting $v_2 = 1$, we have $\mathbf{v} = (-1, 1)$. Since we have only obtained a single eigenvector, we must solve the equation $(A+I)\mathbf{w} = \mathbf{v}$ for $\mathbf{w}$ This gives

$$\left[\begin{array}{cc|c} -3 & -3 & -1 \\ 3 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right].$$

It follows that we have $w_1 + w_2 = \frac{1}{3}$. Setting $w_2 = 0$ we have $w_1 = \frac{1}{3}$ so we have $\mathbf{w} = (\frac{1}{3}, 0)$. We therefore have the general solution to the homogeneous system

$$\mathbf{x}(t) = \left(C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \right) \right) e^{-t}.$$

It follows that the fundamental matrix $\Psi(t)$ is given by

$$\Psi(t) = \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix}$$

so that

$$\begin{aligned} \Psi^{-1}(t) &= \frac{1}{-\frac{1}{3}e^{-2t}} \begin{bmatrix} te^{-t} & -\frac{1}{3}e^{-t} + te^{-t} \\ -e^{-t} & -e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -3te^{t} & e^{t} - 3te^{t} \\ 3e^{t} & 3e^{t} \end{bmatrix}. \end{aligned}$$

It follows that we have

$$\begin{aligned} \Psi^{-1}(s)\mathbf{g}(s) &= \begin{bmatrix} -3se^{s} & e^{s} - 3se^{s} \\ 3e^{s} & 3e^{s} \end{bmatrix} \begin{bmatrix} e^{-s} \\ e^{-s} \end{bmatrix} \\ &= \begin{bmatrix} -6s + 1 \\ 6 \end{bmatrix} \end{aligned}$$

We therefore have

$$\int \Psi(s)^{-1}\mathbf{g}(s)\ ds = \int \begin{bmatrix} -6s + 1 \\ 6 \end{bmatrix} ds = \begin{bmatrix} -3t^2 + t \\ 6t \end{bmatrix}$$

and so

$$\begin{aligned} \Psi(t) \int \Psi(s)^{-1}\mathbf{g}(s)\ ds &= \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix} \begin{bmatrix} -3t^2 + t \\ 6t \end{bmatrix} \\ &= \begin{bmatrix} -3t^2e^{-t} + te^{-t} \\ 3t^2e^{-t} + te^{-t} \end{bmatrix}. \end{aligned}$$

It follows that the general solution is given by

$$\begin{aligned} \mathbf{x}(t) &= \Psi(t)\mathbf{c} + \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\ ds \\ \Longrightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} -3t^2e^{-t} + te^{-t} \\ 3t^2e^{-t} + te^{-t} \end{bmatrix}. \end{aligned}$$