MATH 320, Spring 2013, Assignment 8
Due date: ~~Monday, April 8~~ Wednesday, April 10

Name (printed): _____

UW Student ID Number: _____

Discussion Section: (circle)

Robin Prakash:	301	302	303
Sowmya Acharya:	304 306 307 308		
Raghvendra Chaubey:	352 353 354 355		

Instructions

1. Fill out this cover page **completely** and affix it to the front of your submitted assignment.

2. **Staple** your assignment together and answer the questions in the order they appear on the assignment sheet.

3. Show all the work required to obtain your answers.

4. You are encouraged to collaborate on assignment problems but you must write up your assignment independently. **Copying is strictly forbidden!**

S#	Q#	Mark
4.1	—	/5
4.3	—	/5
4.4	—	/5
4.5	—	/5
—	1	/5
Total:		/25
Bonus:		/2

Vector Spaces, Basis, Matrix Spaces

Suggested problems:

Section 4.1: 1-36
Section 4.3: 1-22
Section 4.4: 1-10, 15-26
Section 4.5: 1-16, 27-30

Problems for submission:

Section 4.1: 21, 24, 28
Section 4.3: 13, 16
Section 4.4: 2, 8, 20
Section 4.5: 6, 12
(Justify your answers for full marks!)

1. (a) Consider an 2×2 matrix A with two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct real-valued eigenvalues λ_1 and λ_2 (i.e. $\lambda_1 \neq \lambda_2$). That is to say, suppose we have

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \text{and} \quad A\vec{v}_2 = \lambda_2 \vec{v}_2.$$

Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. [**Hint:** Use the definition of linear dependence to rewrite $\vec{v}_2$ in terms of $\vec{v}_1$.] Conclude that $\{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$.

   (b) The eigenvectors of a matrix provide a particularly useful basis for operations for which can be represented as linear transformations $\vec{w} = A\vec{v}$. This is because they allow the operation of the matrix (which could be quite complicated) to be simplified to the operation in some set of invariant directions (which we will learn how to identify). It turns out that, once these directions are identified, we can consider the transformation on *any* vector by restricting to just this basis and computing the action component by component rather than all at once.

   To this end, consider the matrix

$$A = \begin{bmatrix} -3 & 8 \\ 0 & 5 \end{bmatrix}.$$

It can be shown that this matrix satisfies the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ for the eigenvalue / eigenvector pairs $\lambda_1 = -3$, $\vec{v}_1 = (1,0)$ and $\lambda_2 = 5$, $\vec{v}_2 = (1,1)$. As expected, the eigenvectors are linearly independent and form a basis for $\mathbb{R}^2$ so that we can write any vector in $\mathbb{R}^2$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Consider the following vectors $\vec{v}$ and determine how *both* $\vec{v}$ and the image vector $\vec{w} = A\vec{v}$ can be written in the coordinates of the basis $\vec{v}_1$ and $\vec{v}_2$:

$$\text{(i) } \vec{v} = (2,1), \quad \text{(ii) } \vec{v} = (0,-2), \quad \text{(iii) } \vec{v} = (5,-3).$$

(In other words, find c_1, c_2, $\tilde{c}_1$, and $\tilde{c}_2$ so that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{w} = \tilde{c}_1\vec{v}_1 + \tilde{c}_2\vec{v}_2$.) Do you notice a pattern?

(c) Suppose A is an $n \times n$ matrix with n distinct eigenvalues and n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. In other words, assume we have

$$A\vec{v}_1 = \lambda_1\vec{v}_1, \quad A\vec{v}_2 = \lambda_2\vec{v}_2, \quad \ldots \quad A\vec{v}_n = \lambda_n\vec{v}_n.$$

Show that if a vector $\vec{v} \in \mathbb{R}^n$ has coordinates $(c_1, c_2, \ldots, c_n)$ with respect to $V = \{\vec{v}_1, \ldots, \vec{v}_n\}$ then $\vec{w} = A\vec{v}$ has coordinates $(c_1\lambda_1, c_2\lambda_2, \ldots, c_n\lambda_n)$ with respect to V.

(d) **Bonus!** Generalize the argument from part (a) to show that any set $V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of eigenvectors corresponding to distinct real-valued eigenvalues is linearly independent. [**Hint:** This can be done directly by generalizing the argument from part (a) to more vectors. The easier method, however, is to perform an induction on n, i.e. use the case for $n = 2$ to show $n = 3$, and $n = 3$ implies $n = 4$, etc.]