MATH 320, WEEK 7:
Matrix Inverses

1 Inverses

Let's reconsider the linear system

$$A\vec{x} = \vec{b}$$

where A is a $n \times n$ matrix and $\vec{x}$ and $\vec{b}$ are n-dimensional vectors.

We recognize that a solution to this expression amounts for solving for $\vec{x}$. We have been able to accomplish this so far by using Gaussian elimination. Now that we have a few matrix algebra operations to work with, we would like to see if we can accomplish the goal of isolating $\vec{x}$ in a more *algebraic* way.

Take the analogy of an algebraic equation in the real numbers:

$$a \cdot x = b.$$

In order to solve for x, the method is simple: we divide both the left-hand and right-hand sides by a to get

$$x = \frac{b}{a}.$$

Now reconsider the linear system $A\vec{x} = \vec{b}$. If we are going to believe we can really develop a comprehensive algebraic system for matrices, we had better believe that it will include an operation like division. It is one of the most basic things we can do! But we have already seen that multiplication is more complicated than simply performing operations component-wise. Division must similarly be overwhelming (and it is!).

The trick to expanding our system of matrix algebra operations to include division is to view division not as an operation in its own right, but to it view as *the inverse of matrix multiplication*. We can be even more explicit than this, is fact. Consider again the algebraic expression $ax = b$ over the real numbers. Rather than treating division as its own operation, we could have written

$$ax = b \implies a^{-1}ax = a^{-1}b \implies 1 \cdot x = a^{-1}b \implies x = \frac{b}{a}.$$

Of course, writing all this in place of a simple division operation with real numbers is grown-inducing. But with matrices we will have no choice! There is no explicitly defined division operation. But we do have an important intuition from the above reasoning. We need to find a matrix (call it A^{-1}) which, when multiplied on the left-hand side of A gives the matrix equivalent of the real number 1 (i.e. the identity matrix I) so that when we multiply out the left-hand side we are just left with $\vec{x}$! This would give the analogous chain of *matrix* equations

$$A\vec{x} = \vec{b} \implies A^{-1}A\vec{x} = A^{-1}\vec{b} \implies I\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}.$$

For example, consider the linear system of equations

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

We can solve this by Gaussian elimination, but we can also search for a matrix such that $A^{-1}A = I$. Fortunately, we will not have to guess what this matrix is—we will be able to find it. For our purposes now, it will be sufficient to check. We can easily verify that the matrix

$$A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

is the matrix we want because

$$A^{-1}A = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the previous sequence of matrix operations, we have that the solution must be

$$A\vec{x} = \vec{b} \implies \vec{x} = A^{-1}\vec{b}$$

so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}.$$

There are a few notes worth making about this process:

- It is important to remember that matrix multiplication in general is *not commutative*. If we perform a matrix multiplication on the *left* of one side of an equation, we have to perform it on the *left* of the other. We would end up with an incorrect answer if we, for instance, performed the operation $\vec{b}A^{-1}$.

- And immediate question is: What if A is not a square matrix? For instance, what if A is a 3×4 matrix. We have already seen linear systems corresponding to 3 equations and 4 unknowns. Can we handle this with this method? The answer, for the purposes of this course is no. We will only consider this method for systems where the number of equations and unknowns are the same.

We can see that the matrix A^{-1} (if we can find it) in that it allows us to perform the algebraic operation of division for matrices (dimension permitting). We should pause, however, to give a formal definition and consider some properties resulting from this definition.

Definition 1.1. *Suppose A is an $n \times n$ matrix. Then A will be called **invertible** if there exists an $n \times n$ matrix B for which $BA = I = AB$. Such a matrix will be called the **inverse** of A and will commonly be denoted A^{-1}.*

We have the following notes of matrice inversion:

- An inverse A^{-1} *does not always exist*! This should not come as a surprise, if we consider the algebraic analogue justifying the introduction of A^{-1}. If we have $ax = b$, we are only justified in writing $x = \frac{b}{a}$ if $a \neq 0$. It will turn out that there is a matrix equivalent to dividing by zero, although it is often carefully concealed within the matrices and will not be obvious from looking at them. Matrices which do not have an inverse will be called *singular*.

- The left inverse of (i.e. B such that $BA = I$) is the same as the right inverse (i.e. C such that $AC = I$).

  Proof. We have $BA = I$ and $AC = I$. Basic operations allow us to write

  $$(BA)C = IC = C \implies B(AC) = C \implies BI = C \implies B = C.$$

  $\square$

- If an inverse exists, it is unique.

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Proof. Suppose there exist B and C so that $BA = I = AB$ and $CA = I = AC$. Then

$$C = CI = C(AB) = (CA)B = IB = B.$$

So the inverse is unique. $\square$

- We have not yet considered how to find matrix inverses A^{-1}. Do not feel overwhelmed if you looked at the form of A^{-1} and wonder how I decided *that* was the matrix to use. We will find out soon enough!

It is (relatively) easy to verify that matrix A and B are inverses of one another. All we have to do is verify that $AB = I$ (or $BA = I$). We also know that inverses are unique, and can be multiplied on either side of the original matrix. That leaves only one small question: how do we *find* inverse matrices? That is to say, given a matrix A, how do we *find* the matrix A^{-1} (if it exists)?

To answer this question, we will start by considering the simplest non-trivial case: 2-by-2 matrices. That is to say, given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

let's try to find a matrix B such that $AB = I$ (we can write this as $BA = I$, but the other order will be more useful). That is to say, let's solve

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This system may look daunting, but there is a very important simplification we can make. If we consider the columns of B and I separately, this is equivalent to the two system of equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These matrix equations are immediate recognizable as the matrix form of a linear system of equations of two equations in two unknowns (b_1 and b_3, and b_2 and b_4, respectively). In other words, we can solve b_1, b_2, b_3, and b_4 (i.e.

we can solve for B) by Gaussian elimination! The corresponding coefficient matrices are

$$\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right].$$

Solving the first system will solve for b_1 and b_3 while the second system will solve for b_2 and b_4.

We could perform these operations directly in this form, but there is (yet another!) simplification we can make. We would notice very quickly doing the separate row-reduction procedures that, in order to achieve the row-reduced echelon form, we will be duplicating the same procedures on the left-hand side of the coefficient matrices. This is because the left-hand sides are *identical*. In order to avoid duplication of arithmetic, we can solve both linear systems *simultaneously* by writing

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right].$$

Now, when we perform row reduction, we will have to remember that the first column to the right of the line solves for b_1 and b_3, while the second column to the right of the line solves for b_2 and b_4.

Fortunately, we can perform this row reduction operation directly. We have

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1'=(1/a)R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2'=R_2-cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2'=(a/(ad-bc))R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bd} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_1'=R_1-(b/a)R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Yikes! That was a crazy amount of work, but there is a silver lining: we will never have to do it again. The values to the right-hand side of the dividing

line correspond to the values of b_1, b_2, b_3, and b_4. It turns out that we can read off the matrix B directly. We have

$$B = A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The key point is that this is the form of the inverse matrix A^{-1} *for any* 2×2 matrix A. In other words, we will never have to perform this hideous calculation ever again for a 2×2 matrix!

There are a few notes worth making:

- The general technique for finding the inverse of a 2×2 matrix is to flip the terms along the main diagonal (i.e. a and d), reverse the sign of the terms along the opposite diagonal (i.e. b and c), and divide by $ad - bc$ (which we will give a formal name soon).

- We can determine from this formula exactly what the condition for invertibility is! The only possible way that a 2×2 matrix could fail to be invertible is when $ad - bc = 0$, since this would result in division by zero. This is a necessary and sufficient condition—if all we are interested in is whether a 2×2 matrix is invertible, all we need to check is $ad - bc$. If this evaluates to anything other than zero, the matrix is invertible and we can find the inverse A^{-1} by the above formula.

- Unfortunately, it is only possible to find a general formula for inverses (in a reasonable amount of time) for 2×2 matrices. It is important to note, however, that the technique for finding inverse for higher dimensional square matrices *is exactly the same*. It just happens that in the 2×2 case we have a general formula that is not overwhelming and can be committed to memory (which saves time!).

Now let's return to the original example. Suppose we wanted to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Suppose we did not know what the inverse was, we could find it by solving for the matrix B according to

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can solve for $b_1, \ldots, b_9$ by performing Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1' = -R_2 \\ R_2' = R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 2R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1' = R_1 + R_2 \\ R_3' = R_3 - R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$R_3' = -R_3 \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\begin{array}{c} R_1' = R_1 - 2R_3 \\ R_2' = R_2 - 3R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right].$$

Following the logic from before, we can read off the inverse matrix as what is left on the right-hand side. We have

$$B = A^{-1} = \left[\begin{array}{ccc} -1 & 1 & 2 \\ -2 & 2 & 3 \\ 1 & 0 & -1 \end{array}\right]$$

which we have already verified is the correct inverse.