Math 320, Spring 2013, Term Test I

Linear Algebra and Differential Equations

Date: Friday, February 22

Lecture Section: 001

SOLUTIONS

Name (printed): ______________________________

UW Student ID Number: ______________________________

Discussion Section: (circle)

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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
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5	/6
Total	/25

1. Definitions and Classification:

[2] (a) Classify the following differential equations according to their order, and whether they are linear / nonlinear, autonomous / nonautonomous, and homogeneous / nonhomogeneous.

i. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \sin(y) = 0$ 2nd order, nonlinear, autonomous, homogeneous.

ii. $\frac{dy}{dx} = \frac{1}{x} + \frac{y}{1-x}$ 1st order, linear, nonautonomous, nonhomogeneous

[1] (b) State a condition on the first-order differential equation $\frac{dy}{dx} = f(x,y)$ which is sufficient to guarantee a unique solution to the corresponding initial value problem $y(x_0) = y_0$ (i.e. a unique solution through the point (x_0, y_0)).

$\frac{\partial f}{\partial y}$ continuous at (x_0, y_0)

[3] 2. True/False:

(a) A first-order (power) homogeneous differential equation (i.e. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$) can be reduced to a separable differential equation by the substitution $v = \frac{y}{x}$.
[$\boxed{\text{True}}$ / False]

(b) The integration factor for the first-order linear differential equation $y' - y = x$ is $\rho(x) = e^x$. [True / $\boxed{\text{False}}$]

(c) A differential equation of the form $M(x,y)\ dx + N(x,y)\ dy = 0$ can always be turned into an exact equation by multiplying by an appropriate integration factor.
[True / $\boxed{\text{False}}$]

3. Slope Fields:

Consider the following differential equation:

$$\frac{dy}{dx} = \frac{1}{y} - y. \tag{1}$$

[2] (a) Show that $y(x) = \pm\sqrt{1 + k\bar{e}^{2x}}$ is a solution of (1) for $k \in \mathbb{R}$.

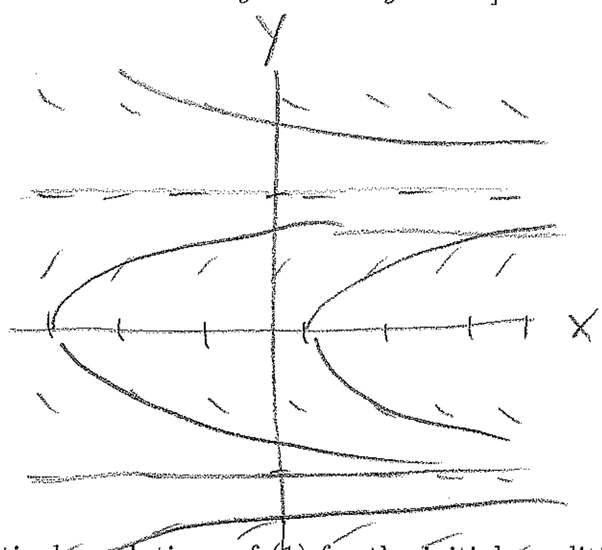
LHS: $\frac{dy}{dx} = \pm \frac{1}{2\sqrt{1+ke^{-2x}}}(-2ke^{-2x}) = \mp \frac{ke^{-2x}}{\sqrt{1+ke^{-2x}}}$

RHS: $\frac{1}{y} - y = \frac{1}{\pm\sqrt{1+ke^{-2x}}} \mp \sqrt{1+ke^{-2x}} = \frac{1-(1+ke^{-2x})}{\pm\sqrt{1+ke^{-2x}}}$

$= \mp \frac{ke^{-2x}}{\sqrt{1+ke^{-2x}}}$

[2] (b) Sketch the slope field over the range $-2 \le x \le 2$, $-2 \le y \le 2$ and, based on this sketch and the solution $y(x)$ in part (a), overlay a few particular solutions. [**Hint:** Be sure to consider values between $y = 0$ and $y = \pm 1$!]

y	$f(x,y)$
-2	1.5
-1	0
$-\frac{1}{2}$	-1.5
0	∞
$\frac{1}{2}$	1.5
1	0
2	-1.5



[2] (c) Determine the particular solutions of (1) for the initial conditions $y(0) = 1/2$ and $y(0) = -1/2$. Comment on these domain of these solutions, making note of any discontinuities in $f(x, y)$. [**Hint:** Consider what happens as $y \to 0$ in $f(x, y)$ and the particular solutions.]

$y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \sqrt{1+ke^{0}} \Rightarrow \frac{1}{4} = 1+k \Rightarrow k = -\frac{3}{4}$ $(+)$

$y(0) = -\frac{1}{2} \Rightarrow -\frac{1}{2} = -\sqrt{1+ke^{0}} \Rightarrow \frac{1}{4} = 1+k \Rightarrow k = -\frac{3}{4}$ $(-)$

$\Rightarrow y(x) = \sqrt{1-\frac{3}{4}e^{-2x}}$ and $y(x) = -\sqrt{1-\frac{3}{4}e^{-2x}}$

Solution does not exist when $1-\frac{3}{4}e^{-2x} < 0$

This corresponds to $y \to 0$ which implies $f(x,y) \to \infty$ (solution breaks down).

4. General Solutions:

Find the general solutions of the following differential equations:

[3] (a) $3x^5y^2 + x^3\dfrac{dy}{dx} = 2y^2$

$$x^3\frac{dy}{dx} = 2y^2 - 3x^5y$$

$$\Rightarrow x^3\frac{dy}{dx} = y^2(2-3x^5) \qquad \boxed{\text{separable}}$$

$$\Rightarrow \frac{1}{y^2}dy = \left(\frac{2-3x^5}{x^3}\right)dx = \left(\frac{2}{x^3} - 3x^2\right)dx$$

$$\Rightarrow -\frac{1}{y} = -\frac{1}{x^2} - x^3 + C = \frac{-1-x^5+Cx^2}{x^2}$$

$$\Rightarrow \boxed{y = \frac{x^2}{1+x^5+Cx^2}}$$

[4] (b) $x^2\dfrac{dy}{dx} + 2xy = 5y^4$

$\boxed{\text{Bernoulli}}$ $$\frac{dy}{dx} + \frac{2}{x}y = \frac{5}{x^2}y^4$$

$$v = y^{1-n} = y^{-3} \Rightarrow y = v^{-\frac{1}{3}} \Rightarrow y^4 = v^{-\frac{4}{3}}$$

$$\frac{dy}{dx} = -\frac{1}{3}v^{-\frac{4}{3}}\frac{dv}{dx}$$

$$\Rightarrow -\frac{1}{3}v^{-\frac{4}{3}}\frac{dv}{dx} + \frac{2}{x}v^{-\frac{1}{3}} = \frac{5}{x^2}v^{-\frac{4}{3}}$$

$$\Rightarrow \frac{dv}{dx} - \frac{6}{x}v = \frac{-15}{x^2} \Rightarrow \frac{1}{x^6}\frac{dv}{dx} - \frac{6}{x^7}v = \frac{-15}{x^8}$$

$$\rho(x) = e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = \frac{1}{x^6} \qquad \Rightarrow \frac{d}{dx}\left[\frac{1}{x^6}v\right] = \frac{-15}{x^8}$$

$$\Rightarrow \frac{1}{x^6}v = \frac{15}{7x^7} + C$$

$$\Rightarrow v = \frac{15}{7x} + Cx^6$$

$$\Rightarrow y = \sqrt[3]{\frac{7x}{15 + Cx^7}}$$

5. Applications:

Consider a filled mixing tank with a volume of 20 gallons. Suppose there is an inflow pipe which pumps in a 0.5 lb/gallon brine (salt/water mixture) at a rate of 2 gallons per minute, and there is an outflow pipe which removes the mixture from the tank at a rate of 2 gallons per minute.

[1] (a) Use the given information to derive a differential equation which models the amount of salt in the tank.

$$\frac{dA}{dt} = [\text{in}] - (\text{out}] = (0.5)(2) - 2\frac{A}{20} = 1 - \frac{A}{10}$$

[3] (b) Find the general solution of the differential equation derived in part (a).

$$\frac{dA}{dt} + \frac{A}{10} = 1 \Rightarrow p(t) = e^{\int \frac{1}{10} t} = e^{\frac{1}{10}t}$$

$$\Rightarrow e^{\frac{1}{10}t}\frac{dA}{dt} + \frac{1}{10}e^{\frac{1}{10}t}A = e^{\frac{1}{10}t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{\frac{1}{10}t}A\right] = e^{\frac{1}{10}t}$$

$$\Rightarrow e^{\frac{1}{10}t}A = 10e^{\frac{1}{10}t} + C$$

$$\Rightarrow A(t) = 10 + Ce^{-\frac{1}{10}t}$$

[2] (c) Suppose there is initially no salt in the tank. How much salt is in the tank after ten minutes? [**Note:** You do not need to evaluate to a decimal value, although, if it helps, $e^{-1} \approx 0.3678794412$.]

$$A(0)=0 \Rightarrow A(0) = 0 = 10 + C \Rightarrow C = -10$$

$$\Rightarrow A(t) = 10 - 10e^{-\frac{1}{10}t}$$

$$A(10) = 10 - 10e^{-1} = 10 - 10(0.367879)$$

$$\approx 10 - 3.678\ldots$$

$$\approx 6.321\ldots$$