Math 320, Spring 2013, Term Test I

Linear Algebra and Differential Equations

Date: Friday, February 22

Lecture Section: 004

SOLUTIONS

Name (printed): ______________________________

UW Student ID Number: ______________________________

Discussion Section: (circle)

Robin Prakash:	**301**	**302**	**303**	
Sowmya Acharya:	**304**	**306**	**307**	**308**
Raghvendra Chaubey:	**352**	**353**	**354**	**355**

Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
Page	Mark
2	/6
3	/6
4	/7
5	/6
Total	/25

1. Definitions and Classification:

[2] (a) Classify the following differential equations according to their order, and whether they are linear / nonlinear, autonomous / nonautonomous, and homogeneous / nonhomogeneous.

i. $\frac{d^2y}{dx^2} - y^2 = y$ Second-order, nonlinear, autonomous, homogeneous

ii. $\frac{dy}{dx} = \sin(x) + xy$ First-order, linear, nonautonomous, nonhomogeneous.

[1] (b) State the condition required for the differential equations $M(x, y)\, dx + N(x, y)\, dy = 0$ to be exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

[3] 2. True/False:

(a) A Bernoulli differential equation (i.e. $\frac{dy}{dx} + P(x)y = Q(x)y^n$) can be reduced to a first-order linear differential equation by the substitution $v = y^{1-n}$. [True / False]

(b) The integration factor for the first-order linear differential equation $y' + \frac{1}{x}y = \sin(x)$ is $\rho(x) = x$. [True / False]

(c) A sufficient condition for the first-order differential equation $\frac{dy}{dx} = f(x, y)$ to have a unique solution through (x, y) is that $f(x, y)$ is continuous at (x, y). [True / False]

3. Slope Fields:

Consider the following differential equation:

$$\frac{dy}{dx} = \sqrt{1-y^2}, \quad -1 \le y \le 1. \tag{1}$$

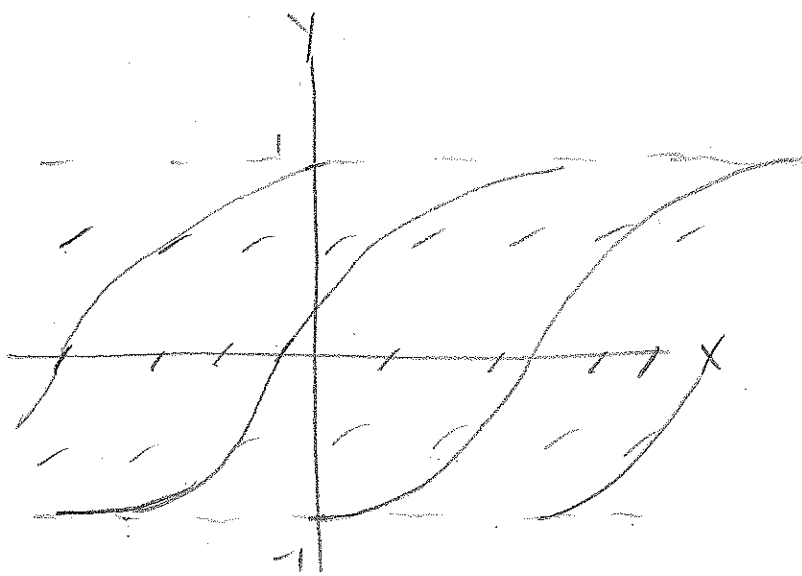
[2] (a) Show that $y(x) = \sin(x - C)$ for $C - \pi/2 \le x \le C + \pi/2$ is a solution to (1) for all $C \in \mathbb{R}$.

LHS: $\frac{dy}{dx} = \cos(x-C)$

RHS: $\sqrt{1-y^2} = \sqrt{1-\sin^2(x-C)}$
$= \sqrt{\cos^2(x-C)} = |\cos(x-C)|$
$= \cos(x-C)$
(since $\cos(x-C) \ge 0$ for $C - \frac{\pi}{2} \le x \le C + \frac{\pi}{2}$)

[2] (b) Sketch the slope field and, based on this sketch and the solution $y(x)$ in part (a), overlay a few particular solutions.

y	$f(x,y)$
-1	0
-0.5	$\frac{\sqrt{3}}{2}$
0	1
0.5	$\frac{\sqrt{3}}{2}$
1	0



[2] (c) Determine the particular solution of (1) for the initial conditions $y(0) = 0$. Comment on the uniqueness of solutions at the limits of the solution, i.e. consider what happens at $x = C - \pi/2$ and $x = C + \pi/2$. [**Hint:** Try to find another solution to (1)!]

$y(0)=0 \Rightarrow 0 = \sin(-C) \Rightarrow C = 0.$
$\Rightarrow y(x) = \sin(x)$

Since $y=1$ and $y=-1$ are also solutions, it follows that uniqueness breaks down at these points (i.e. at $x = C \pm \frac{\pi}{2}$)

4. General Solutions:

Find the general solutions of the following differential equations:

[3] (a) $(1+x)^2 \dfrac{dy}{dx} = (1+y)^2$ $\boxed{\text{Separable}}$

$$\int \frac{1}{(1+y)^2}\,dy = \int \frac{1}{(1+x)^2}\,dx$$

$$\Rightarrow -\frac{1}{1+y} = -\frac{1}{1+x} + C$$

$$\Rightarrow \frac{1}{1+y} = \frac{1+C(1+x)}{1+x}$$

$$\Rightarrow 1+y = \frac{1+x}{1+C(1+x)} \Rightarrow y(x) = \frac{1+x}{1+C(1+x)} - 1$$

[4] (b) $x\dfrac{dy}{dx} = y + 2\sqrt{xy}$ $\boxed{\text{power homogeneous}}$

$$v = \frac{y}{x} \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow x\left(v + x\frac{dv}{dx}\right) = xv + 2\sqrt{x(xv)}$$

$$\Rightarrow x\left(v + x\frac{dv}{dx}\right) = xv + 2x\sqrt{v}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + 2\sqrt{v}$$

$$\Rightarrow x\frac{dv}{dx} = 2\sqrt{v} \Rightarrow \frac{1}{2\sqrt{v}}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \sqrt{v} = \ln|x| + C$$

$$\Rightarrow v = (\ln|x| + C)^2$$

$$\Rightarrow \frac{y}{x} = (\ln|x| + C)^2 \Rightarrow y(x) = x(\ln|x| + C)^2$$

5. Applications:

Assume that a population (denoted P) grows at a rate proportional to its own population when the population size is small (i.e. proportional to P) but encounters a third-power crowding term when the population is large (i.e. proportional to P^3). After simplifying, rescaling and factoring, this gives rise to the growth model

$$\frac{dP}{dt} = P(1 - P^2). \tag{2}$$

[5] (a) Find the general solution of (2). [**Hint:** The equation is separable, but there is an easier solution method.]

$\frac{dP}{dt} = P - P^3 \Rightarrow \frac{dP}{dt} - P = -P^3$ $\boxed{\text{Bernoulli}}$

$\Rightarrow v = P^{1-n} = P^{-2} \Rightarrow P = v^{-\frac{1}{2}} \Rightarrow P^3 = v^{-\frac{3}{2}} \Rightarrow \frac{dP}{dt} = -\frac{1}{2} v^{-\frac{3}{2}} \frac{dv}{dt}$

$\Rightarrow -\frac{1}{2} v^{-\frac{3}{2}} \frac{dv}{dt} - v^{-\frac{1}{2}} = -v^{-\frac{3}{2}}$

$\Rightarrow \frac{dv}{dt} + 2v = 2 \Rightarrow e^{2t} \frac{dv}{dt} + 2e^{2t} v = 2e^{2t}$

$\rho(t) = e^{\int 2\,dt} = e^{2t}$ $\Rightarrow \frac{d}{dt}[e^{2t} v] = 2e^{2t}$

$\Rightarrow e^{2t} v = e^{2t} + C \Rightarrow v = 1 + Ce^{-2t}$

$\Rightarrow P^{-2} = 1 + Ce^{-2t} \Rightarrow P = \frac{1}{\sqrt{1 + Ce^{-2t}}}$

[1] (b) What is the limiting value of the population? [**Hint:** Take the limit as $t \to \infty$.]

$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{1}{\sqrt{1 + \underbrace{Ce^{-2t}}_{\to 0}}} = 1$