Math 320, Spring 2013, Term Test II

Linear Algebra and Differential Equations

Date: Friday, April 12

Lecture Section: 001

Name (printed): ______________________

UW Student ID Number: ______________________

Discussion Section: (circle)

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Solutions

Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

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1. Definitions:

[1] (a) State the definition of what it means for a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ to be linearly independent.

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$
has only the solution $c_1 = c_2 = \cdots = c_n = 0$.

[1] (b) Give the definition of what it means for V to be a *vector space.*

1) for all $c \in \mathbb{R}$, and $\vec{v} \in V$ we have $c\vec{v} \in V$.
2) for all $\vec{v}, \vec{w} \in V$, we have $\vec{v} + \vec{w} \in V$.

[1] (c) State the Rank-Nullity Theorem.

Let A be an $m \times n$ matrix. Then
$\text{rank}(A) + \text{nullity}(A) = n$

[3] 2. True/False:

(a) If an $n \times n$ matrix A has a left inverse B then the right inverse is the adjoint of B. [True / False]

(b) An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. [True / False]

(c) Every basis of a 4-dimensional subspace of $\mathbb{R}^7$ has 4 vectors in it. [True / False]

~~(d) If an $n \times n$ matrix has less than n distinct real eigenvalues then it must have at least one complex eigenvalue. [True / False]~~

3. Gaussian Elimination:

[4] (a) Write the following linear system as the matrix equation $A\vec{x} = \vec{b}$ then determine values of $k \in \mathbb{R}$ for which the system has (i) a unique solution; (ii) no solution; and (iii) infinitely many solutions.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ -2x_1 - 3x_2 + x_3 &= 1 \\ -3x_1 + x_2 + kx_3 &= -4. \end{aligned} \quad \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -3 & 1 & k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ -2 & -3 & 1 & 1 \\ -3 & 1 & k & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & k-3 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & k+4 & 0 \end{array}\right]$$

$\Rightarrow$ One solution if $k \neq -4$
Infinitely many solutions if $k = -4$
No solution $\Rightarrow$ not possible.

[2] (b) Use the fact that

$$A = \begin{bmatrix} -3 & 1 & 4 \\ -1 & 3 & -2 \\ 3 & -1 & -5 \end{bmatrix} \implies A^{-1} = \frac{1}{8}\begin{bmatrix} -17 & 1 & -14 \\ -11 & 3 & -10 \\ -8 & 0 & -8 \end{bmatrix}$$

to solve the linear system

$$\begin{aligned} -3x_1 + x_2 + 4x_3 &= 1 \\ -x_1 + 3x_2 - 2x_3 &= 1 \\ 3x_1 - x_2 - 5x_3 &= 1. \end{aligned}$$

$$\begin{bmatrix} -3 & 1 & 4 \\ -1 & 3 & -2 \\ 3 & -1 & -5 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (A\vec{x} = \vec{b})$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -17 & 1 & -14 \\ -11 & 3 & -10 \\ -8 & 0 & -8 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (\vec{x} = A^{-1}\vec{b})$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -30 \\ -18 \\ -16 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -15 \\ -9 \\ -8 \end{bmatrix}$$

4. Determinants and Inverses:

[4] Determine the adjoint of the following matrix and use it to find A^{-1} (part marks will be awarded for solving for A^{-1} by another method):

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\det(A) = -(-1)\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = (1)(2)(-1) = -2$$

$A_{11} = -3 \quad A_{12} = 2 \quad A_{13} = 2$

$A_{21} = -1 \quad A_{22} = 0 \quad A_{23} = 0$

$A_{31} = -3 \quad A_{32} = 0 \quad A_{33} = -2$

$$\Rightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix} -3 & -1 & -3 \\ 2 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

5. Matrix Spaces:

[3] Consider the following matrix A and its row-reduced echelon matrix (to the right):

$$A = \begin{bmatrix} 1 & 1 & 3 & -2 \\ 2 & 0 & -4 & 1 \\ 1 & -1 & -7 & 0 \\ -1 & 2 & 12 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

State a basis for the following vector spaces:

(a) The row space of A: $\{(1,0,-2,0), (0,1,5,0), (0,0,0,1)\}$

(b) The column space of A: $\{(1,2,1,-1), (1,0,-1,2), (-2,1,0,7)\}$.

(c) The null space of A: $\{(2,-5,1,0)\}$.

6. Eigenvalues/Eigenvectors:

[4] Determine the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & -4 \\ -2 & -1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ -2 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 8$$
$$= \lambda^2 - 9 = (\lambda - 3)(\lambda + 3) = 0.$$
$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = -3$$

$$\lambda_1 = 3 \Rightarrow \left[\begin{array}{cc|c} -2 & -4 & 0 \\ 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \Rightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7. Theory:

[2] Suppose $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^n$ are pairwise linearly independent (i.e. we have $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$ are linearly independent). Prove or disprove the claim that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. [**Hint:** To prove, you must show the claim holds; to disprove, find a counter example.]

NOT TRUE! Even for vectors in $\mathbb{R}^2$, we can see that $\vec{v}_1 = (1,0)$, $\vec{v}_2 = (0,1)$ and $\vec{v}_3 = (1,1)$ are pairwise linearly independent but $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.