Math 320, Spring 2013, Term Test II

Linear Algebra and Differential Equations

Date: Friday, April 12

Lecture Section: 004

SOLUTIONS

Name (printed): ____________________

UW Student ID Number: ____________________

Discussion Section: (circle)

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Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

FOR EXAMINERS' USE ONLY	
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1. Definitions:

[1] (a) State the definition of what it means for a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ to be *linearly dependent.*

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$ has a solution other than $c_1 = c_2 = \ldots = c_n = 0$.

[1] (b) State the Rank-Nullity Theorem.

Let A be an $m \times n$ matrix. Then: $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$

[1] (c) State a necessary and sufficient condition for an $n \times n$ matrix to be invertible.

$\det(A) \neq 0$ (or $\operatorname{rank}(A) = n$)

[3] 2. True/False:

(a) If an $n \times n$ matrix A is invertible then the inverse is unique. [True / False]

(b) If an $n \times n$ matrix has fewer than n real distinct eigenvalues it must have a complex eigenvalue. [True / False]

(c) Every basis of a 2-dimensional subspace of $\mathbb{R}^5$ has $5 - 2 = 3$ vectors in it. [True / False]

3. Gaussian Elimination:

[4] (a) Write the following linear system as the matrix equation $A\vec{x} = \vec{b}$ then determine values of $k \in \mathbb{R}$ for which the system has (i) a unique solution; (ii) no solution; and (iii) infinitely many solutions.

$$\begin{aligned} 2x_1 - x_2 \quad &= 2 \\ -x_1 + x_2 + kx_3 &= 3 \\ x_2 + 2x_3 &= -1. \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & k \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ -1 & 1 & k & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -k & -3 \\ 0 & 1 & 2k & 8 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -k & -3 \\ 0 & 1 & 2k & 8 \\ 0 & 0 & 2k-2 & -9 \end{array}\right] \Rightarrow$$

No solution if $k=1$
One solution if $k \neq 1$
Infinitely many solutions not possible.

[2] (b) Use the fact that

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 3 & -2 \\ -1 & -1 & -2 \end{bmatrix} \implies A^{-1} = \frac{1}{4}\begin{bmatrix} -4 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & -2 \end{bmatrix}$$

to solve the linear system

$$\begin{aligned} -x_1 + \ x_2 + \ \ x_3 &= 3 \\ x_1 + 3x_2 - 2x_3 &= -2 \\ -x_1 - \ x_2 - 2x_3 &= 5. \end{aligned}$$

$$\begin{bmatrix} -1 & 1 & -2 \\ 1 & 3 & -2 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \qquad (A\vec{x} = \vec{b})$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \qquad (\vec{x} = A^{-1}\vec{b})$$

$$= \frac{1}{4}\begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$$

4. Determinants and Inverses:

[4] Determine the adjoint of the following matrix and use it to find A^{-1} (part marks will be awarded for solving for A^{-1} by another method):

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 3 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$\det(A) = -(3)(2)(3) = -18$

$A_{11} = 0 \quad A_{12} = 0 \quad A_{13} = -6$

$A_{21} = -6 \quad A_{22} = -6 \quad A_{23} = -2 \quad \Rightarrow A^{-1} = \frac{-1}{18}\begin{bmatrix} 0 & -6 & -9 \\ 0 & -6 & 0 \\ -6 & -2 & -3 \end{bmatrix}$

$A_{31} = -9 \quad A_{32} = 0 \quad A_{33} = -3$

5. Matrix Spaces:

[3] Consider the following matrix A and its row-reduced echelon matrix (to the right):

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 & -1 \\ -1 & 2 & -2 & -2 & 0 \\ 1 & -2 & 4 & 8 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

State a basis for the following vector spaces:

(a) The row space of A: $\{(1,-2,0,-4,2),(0,0,1,3,-1)\}$.

(b) The column space of A: $\{(1,-1,1),(3,-2,4)\}$.

(c) The null space of A:

$\{(2,1,0,0,0),(4,0,-3,1,0),(-2,0,1,0,1)\}$

6. Eigenvalues/Eigenvectors:

[4] Determine the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 2 \\ -12 & 5-\lambda \end{vmatrix} = (-5-\lambda)(5-\lambda) + 24$$
$$= \lambda^2 - 25 + 24 = \lambda^2 - 1 = 0$$
$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = -1$$

$$\lambda_1 = 1 \Rightarrow \left[\begin{array}{cc|c} -6 & 2 & 0 \\ -12 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda_2 = -1 \Rightarrow \left[\begin{array}{cc|c} -4 & 2 & 0 \\ -12 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

7. Theory:

[2] Suppose $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^n$ are pairwise linearly independent (i.e. we have $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$ are linearly independent). Prove or disprove the claim that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. [**Hint:** To prove, you must show the claim holds; to disprove, find a counter example.]

NOT TRUE! Even for vectors in $\mathbb{R}^2$, we can see that $\vec{v}_1 = (1, 0)$, $\vec{v}_2 = (0, 1)$ and $\vec{v}_3 = (1, 1)$ are pairwise linearly independent but $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.