MATH 320, WEEK 1:
Introduction, Definitions & Review

1 Introduction

This course is intended to be an introduction to *differential equations* and *linear algebra*.

While these two topics are often treated separately, there are several advantages to learning them together. The most obvious advantage is that linear algebra is incredibly useful in analysing differential equations. It is the linear algebra toolbox that will allow us to formalize and systematize the study of ordinary differential equations. It simply cannot be escaped: to properly study differential equations, one needs to understand linear algebra.

That said, linear algebra is a huge discipline in its own right. It has applications on the mathematical side of practically every applied science—from computer science, to probability theory, to economics. The study of this topic given in this course will be necessarily abbreviated and tailored to the study of differential equations in particular, but it will also be a sufficient basis to understand the applications in other disciplines.

The course will be broken down into approximately the following chunks:

- First five weeks: **Differential equations**
  - Basic definitions, notation, theorems, and applications.
  - We will investigate and solve selected differential equations using a variety of special solution methods.
  - It should be noted and expected that the methods we will encounter during this section of the course will be presented on a very case-by-case basis. That is to say, we will encounter a number of *tricks* which may not be—and are not intended to be—obvious or necessarily related to the previous tricks we have encountered. Nevertheless, these tricks are very important to understanding approaches taken to analysing and solving differential equations, and several of the classes of differential equations (e.g. separable, first-order linear) will be recurrent.

  Second five weeks: **Linear algebra**

- Basic definitions, notation, theorems, and applications (again!).
- This will be a standard introduction to the basic elements of linear algebra: matrices and vectors, matrix operations, inverses and determinants, linear independent and bases, eigenvalues and eigenvectors.

Third five weeks: **Linear systems of differential equations**

- Combination of the two previous topics!
- Unsurprisingly, the tools develop to this point are not independent, and we will spend the final four to five weeks combining them. We will revisit a few examples for the first five weeks, but we will primarily forge ahead into the analysis of linear systems of differential equations. The tools developed here are further useful in the study of non-linear systems of differential equations (which is investigated in Math 415).

This is all well and good, but we might still be wonder what, after all, a *differential equation* is. And perhaps just as importantly, we might be wondering why we are interested in them. So, in order to set the stage for the central study of this course, let's ask the following questions:

1. What is a differential equation?
2. How do differential equations arise in practice?

The simple answer to the first question is that **a differential equation is any equation (i.e. algebraic expression) which involves functions and their derivatives**. We can find examples which take full advantage of the generality of this expression, and dream up examples with fourth or fifth order derivatives, or examples which have complicated variable dependences, but we need not do so. In fact, differential equations can be formulated very simply. For example, we all know from basic calculus that the expression

$$y = \sin(x)$$

gives rise to the derivative

$$\frac{dy}{dx} = \cos(x). \tag{1}$$

We have grown quite accustomed to working in this direction. If we look at this for just a moment, however, we notice that (1) exactly fits into the definition we have just given. It is a differential equation! (Although a rather trivial one.)

We can also get a sense from this example of what it means to *solve* a differential equation. **A solution to a differential equation is any function which satisfies the equation.** We can clearly see (by definition) that $y(x) = \sin(x)$ satisfies the left-hand and right-hand side of (1). (In fact, it is not the only such choice, but we will get to that later.) This would have been obvious even if we had only been given (1) because we could *integrate* the expression to recover the function $y(x)$ (since the First Fundamental Theorem of Calculus guarantees that integration *undoes* differentiation).

It might appear at this point as though I have led us into a circle of sorts. If all we are doing is taking expressions with derivatives and being asked *go backwards*, we probably immediately have a voice in the back of our heads telling is that we already learned that process. It was called *integration* and was the focus of all of our previous calculus courses (alongside differentiation).

As we will see, integration is indeed a very important process in the study of differential equations. All of the integration techniques considered in previous courses (integration by substitution, integration by parts, trigonometric substitution, integration of rational functions, etc.) will be very important in understanding and solving differential equations. **These topics will be considered background knowledge and will not be reviewed in this course!** If you struggled with those topics in your previous calculus courses, it is very important to review them as soon as you can. They will be very important throughout this course!

That said, it turns out that integration is not *sufficient* for solving differential equations. To see why this the case, let's move on to the second question: how do differential equations arise?

The answer is that many real-world applied phenomena are understood by their *rates* of movement (or rate of rate of movement, i.e. acceleration, etc.). The most readily available example is Newton's second law of motion, which says that the force exerted on an object is equal to its mass times it acceleration, i.e.

$$F = ma. \tag{2}$$

We all know that an objections acceleration is the rate of change (i.e. derivative) of its velocity, which is the rate of change (i.e. derivative) of the object's position, so that the acceleration is the second derivative of the object's position. In other words, we have

$$ma = m\frac{d^2x}{dt^2}.$$

That clarifies the right-hand side of (2), but what about the left-hand side? Depending on the application, different terms are used to represent the forces acting on a body. One simple assumption, which is used commonly in simple models of springs (via Hooke's law) or pendulums (as a result of gravity) is to assume that there is a **restoring force** proportional to the object's distance from its resting position. This is common represented as $F(x) = -kx$ where $k > 0$. (Notice that if $x > 0$, i.e. if the object is to the right of its resting position, then there is a restoring force pushing to the left; conversely, if $x < 0$, i.e. if the object is to the left of its resting position, then there is a restoring force pushing to the right.)

Combining this together into a equation via (2), we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{3}$$

This is certainly a differential equation (it involves the function $x(t)$ and one of its derivatives, in this case the second derivative) but it **cannot be solved directly by integration**. To see why, recall that in order to integrate we need to have a function of the independent variable (in this case, t). In this case, however, we have the unknown function $x(t)$. We cannot integrate over t because we do not know what $x(t)$ is! In fact, determining what $x(t)$ is is exactly what we are trying to ascertain.

Nevertheless, we can still sensibly ask the question of what a solution to (3) might look like. All we are asking for is to find a function $x(t)$ which satisfies the expression. It should not take much convincing that there are several options. The easiest to check are $x_1(t) = \sin\left(\sqrt{\frac{k}{m}}t\right)$ and $x_2(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$. In fact, any solution of the form

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants will work. These solutions were not obtained using integration, however (although we will not get to the general method which was used for a few weeks yet).

There are many other examples of simple differential equations which arise from the sciences which cannot be solved directly by integrating, including:

- **Exponential growth (populations)**

$$\frac{dP}{dt} = rP,$$

- **Logistic growth (populations)**

$$\frac{dP}{dt} = rP(K - P),$$

- **Newton's law of cooling**

$$\frac{dT}{dt} = k(T_{ext} - T),$$

- **Restoring plus friction (second-order linear)**

$$\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0.$$

This gives us some sense of the kind of question we are going to be interested in during this course. We are going to be interested in the following questions:

1. Given a differential equation, is there a solution (i.e. a function which satisfies the expression)? And if so, how can we find it?

2. What other kind of applications give rise to differential equations, and what do those differential equations look like?

3. How do we interpret solutions to differential equations in the context of the governing equations and/or the original physical motivation for them?

2 Notation

It is important first of all to clarify the different notations and terminologies which will be used through the differential equations portion of this course.

The first thing to recognize is that derivatives can and will be represented in a number of different manners. We will have the following equivalent representations for first-order derivatives:

$$\frac{dy}{dx} = y'(x) = \dot{y}.$$

Similarly, second-order derivatives will have the equivalent representations:

$$\frac{d^2y}{dx^2} = y''(x) = \ddot{y}.$$

Where space is no concern, it will be common to use the long-form notation first presented. When many derivatives are present and the independent variable (i.e. x in the above expressions) is assumed, we will favour the latter 'dot' expression. The general notation for an nth-order derivative will be:

$$\frac{d^ny}{dx^n} = y^{(n)}(x).$$

This will allow us to write the notion of what a differential equation in a standard way. In general, the differential equation we will be considering in this class can be written

$$f(x, y(x), y'(x), \ldots y^{(n)}(x)) = 0.$$

A differential equation is said to be in **standard** or **normal form** if it is written

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

where n is the highest-order derivative appearing in the expression.

Example 1: The first-order differential equation

$$xy + \frac{dy}{dx} = 0$$

can be rewritten in the normal form

$$\frac{dy}{dx} = -xy$$

by solving for the first-order derivative. By contrast, the first-order differential equation

$$y - \frac{dy}{dx} \sin\left(\frac{dy}{dx}\right) = 0$$

cannot be rewritten in normal form since the first-order derivative cannot be isolated.

3 Classification of Differential Equations

Many of the tricks we develop for analyzing and solving differential equations over the next four to five weeks will depend on which broad classification of differential equation they belong to. It is important, therefore, to get the distinctions between these kinds of differential equations understood as soon as possible.

Consider the differential equation

$$f(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0. \tag{4}$$

We will say that (4) is:

1. n^{th} **order** (or n^{th} **degree**) if the highest order derivative appearing in (4) is n^{th} order.

2. **Linear** if f is linear in $y(x)$ and all of its derivatives. We can write linear differential equations as

$$\begin{aligned} A_n(x)y^{(n)}(x) + A_{n-1}(x)y^{(n-1)}(x) + \cdots \\ \cdots + A_1(x)y'(x) + A_0(x)y(x) = g(x) \end{aligned} \tag{5}$$

   where the $A_i(x)$, $i = 0, \ldots, n$, and $g(x)$ are allowed to be non-linear in the independent variable x. Otherwise, we will say that (4) is **non-linear**.

3. **Autonomous** if f does not depend explicitly on x, i.e. if

$$f(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = f(y(x), y'(x), y''(x), \ldots, y^{(n)}(x)).$$

   Otherwise, we will say (4) is **non-autonomous**.

4. **Homogeneous** if f does not have any terms which do not depend on $y(x)$ and its derivatives, i.e. if

$$f(x, 0, 0, \ldots, 0) = 0.$$

   Otherwise, we will say (4) is **non-homogeneous**.

Notes:

1. Equations of the form (4) are called **ordinary differential equations** because they depend on derivatives with respect to only a single independent variable (x, in our case). This distinguishes them from

partial differential equations which are equations involving derivatives with respect to *two or more* independent variables. The non-dimensionalized heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is a standard example of a partial differential equation (because there are derivatives which depend on t and x). (We will consider only ordinary differential equations in this course but it is still important to be able to distinguish between ordinary and partial differential equations!)

2. Linearity excludes terms like y^2, $\sin(\dot{y})$, or even $y \cdot \dot{y}$. Basically, wherever an y or any of its derivatives appears in the equation, it must be separated from any other y terms. **It is, however, allowed to have terms with x's attached to it!** For example, a term like $y \sin(x)$ does not violate linearity of the DE even though the term $\sin(x)$ is not linear in x.

3. Autonomous systems do not depend explicitly on the independent variable. When the independent variable is time, these kind of differential equations commonly arise in physical systems (e.g. gravitation laws, electrical/magnetic force fields, pendulums swinging, etc.).

4. Homogeneous systems may depend on the independent variable (in our case, x) but have no "stray" terms involving it. In systems where the independent variable is time, these stray terms typically correspond to some sort of external time-dependent forcing (e.g. shaking a pendulum, imposing an electrical current, etc.).

Examples: Classify the following differential equations according to their order, and whether they are ordinary or partial, linear or non-linear, autonomous or non-autonomous, homogeneous or non-homogeneous.

$$\text{(a)}\ \frac{dy}{dx} = \sin(x)y(x), \quad \text{(b)}\ \frac{d^2y}{dx^2} + \frac{1}{y(x)} = 0, \quad \text{(c)}\ \frac{d^2y}{dx^2} + y(x) = x^2$$

$$\text{(d)}\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad \text{(e)}\ \frac{dy^3}{dx^3} - \frac{dy}{dx}y(x) = e^x.$$

Solution: We have that

- (a) is a linear, non-autonomous, homogeneous, first-order ordinary differential equation,
- (b) is a non-linear, autonomous, homogeneous, second-order ordinary differential equation,
- (c) is a linear, non-autonomous, non-homogeneous, second-order ordinary differential equation,
- (d) is a linear, autonomous, homogeneous, first-order partial differential equation (extending the definitions to partial differential equations),
- (e) is a non-linear, non-autonomous, non-homogeneous, third-order ordinary differential equation.