MATH 320, WEEK 2:
Slope Fields, Uniqueness of Solutions, Initial Value Problems, Separable Equations

1 Slope Fields

We have seen what a differential equation is (relationship with a function and its derivatives), what it means to have a solution (a function which satisfies the relationship), and how differential equations might arise when modeling real-world phenomena (Newton's second law $F = ma$, population growth, etc.). But how do we *interpret* the solution of a differential equation in terms of either the differential equation itself, or the process it is representing?

The simplest answer to this question is to look at the differential equation itself. Let's consider a general first-order differential equation of the form

$$\frac{dy}{dx} = f(x,y). \tag{1}$$

By definition, a function $y(x)$ which *solves* (1) is a function which satisfies the differential equation, i.e. we are looking for a function for which the derivative at a point (x, y) is equal to the given function $f(x, y)$ at (x, y). This is the *algebraic* interpretation of a solution, but it also naturally leads to a *geometrical* interpretation by noticing that the derivative represents the *slope* of the function at a given point (x, y) in the (x, y)-plane. We now have the following intuition about solutions of (1):

At every point (x, y) in the (x, y)-plane, a solution $y(x)$ of (1) must lie tangent to the straight line with slope $f(x, y)$!

This suggests the following interpretative trick:

1. Sample a series of points in the (x, y)-plane and determine the value of $f(x, y)$ at these points.

2. Plot short lines which have the slope of the function $f(x, y)$ at those points. (Positive slope for $f(x, y) > 0$, negative slope for $f(x, y) < 0$, steeper lines for larger values, etc.)

This is called the **slope field** for the system (1) and can be very useful for interpreting solutions of the expression. There are a few notes worth making:

- Slope fields will provide the motivation for *numerical methods*, and in particular *Euler's method*, which we will study in a few weeks. For now, however, it is enough to know how slope fields and solutions fit together.
- A necessary ingredient for generating slope field diagrams is that we are only considering *first-order derivatives*, since we know these derivatives correspond to the slopes of the solution function at the given point. Many differential equations, however, are *not* first-order (e.g. differential equations arising from Newton's second law, which are second-order). Later in the course, we will develop a method for looking at higher-order differential equations as a *system* of first-order differential equations. This will allow us to carry over the intuition offered by slope fields, although at the expense of increasing the dimension of the system.

Example 1: Show that $y(x) = ke^x$ is a solution of $\dfrac{dy}{dx} = y$ for all $k \in \mathbb{R}$. Draw the slope field in the (x, y)-plane and plot a few solutions.

Solution: We have trivially that

$$\frac{dy}{dx} = \frac{d}{dx}\left[ke^x\right] = ke^x = y$$

It only remains to interpret this result.

To draw the slope field, we notice that $f(x, y) = y$ implies that the slopes are independent of x (the system is autonomous). This means we need only consider the value of y. For points with a large y value, the slope will be very steep and positive; for points with a small positive y value, the slope will be shallow and positive. Similar arguments hold for negative y values. In the end, we have the picture give in Figure 1. We can see that any solution $y(x) = ke^x$, $k \in \mathbb{R}$, satisfies the intuition that it lies tangent to the slope field at every point.

Example 2: Show that $y(x) = \tan(x + C)$ is a solution of $\dfrac{dy}{dx} = 1 + y^2$ for all $C \in \mathbb{R}$. Draw the slope field in the (x, y)-plane and plot the solution.

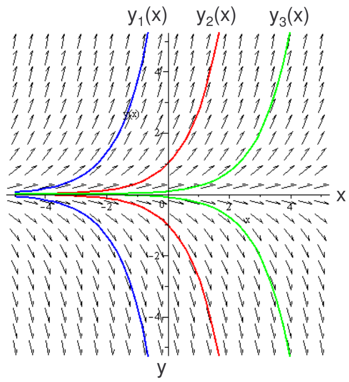

Figure 1: Slope field of $\frac{dy}{dx} = y$ with the solutions $y_1(x) = (1/10)e^x$, $y_2(x) = e^x$, $y_3 = 10e^x$, etc., overlain. Any solution of the form $y(x) = ke^x$, $k \in \mathbb{R}$, satisfies the differential equation.

Solution: On the left-hand side, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan(x+C)\right] = \sec^2(x+C).$$

On the right-hand side, we have

$$1 + y^2 = 1 + \tan^2(x+C) = \sec^2(x+C).$$

It follows that $y(x) = \tan(x+C)$ is a solution.

To draw the slope field, we have $f(x,y) = 1 + y^2$ and notice, once again, that x does not factor in the slopes. We have that the arrows are very steep and positive for very large y, very steep and *positive* for very negative y, and have a minimal steepness of one when $y = 0$ (i.e. along the x-axis). If we are careful, we should arrive at a picture looking like Figure 2.

2 Initial-Value Problems

We have seen that differential equations can, in general, given rise to *multiple* solutions. This should be reasonable disconcerting at first glance. After all, we imagine differential equations as representing some sort of physical phenomenon, and when we throw a projectile, or release a pendulum, or connect an electrical circuit, we do not observe *multiple* solutions. We observe

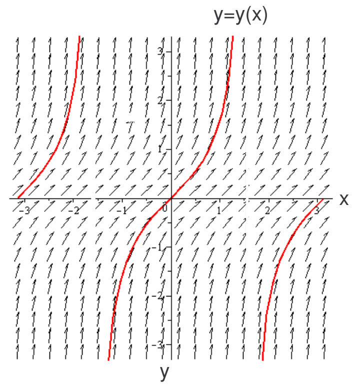

Figure 2: Slope field of $\dfrac{dy}{dx} = 1 + y^2$ with the solution $y(x) = \tan(x)$ overlain.

exactly one. So how do we resolve the mathematical peculiarity of multiple solutions with the physical observation that only one thing can happen at a time?

The answer is that we define the differential equation *together* with the relevant *initial conditions*.

Definition 2.1. *The* ***initial-value problem*** *(IVP) associated with a first-order differential equation is the problem of solving*

$$\frac{dy}{dx} = f(x, y), \qquad \textit{subject to} \quad y(x_0) = y_0$$

where $x_0, y_0 \in \mathbb{R}$.

There are a few notes worth making here:

- The terminology *initial-value* is chosen to reflect the reality that we are usually interested in centering the problem at zero (i.e. setting $x_0 = 0$). In problems where time is the independent variables, we have $t_0 = 0$, which is truly the initial value. We can, however, choose x_0 equal to another value (e.g. conditions like $y(3) = -7$ or $y(-1) = 10$).

- The initial-value problem corresponds to picking the single trajectory which goes through the point (x_0, y_0) in the slope field diagram! We now know exactly how to fill out the slope field diagram with solutions.

- In general, we need as many initial conditions as we have constants in the general solution. For second-order differential equations, we will typically need *two* initial conditions, one on the variable itself, and one on the derivatives. For instance, for gravitational force problems where $x(t)$ is the height of an object, we need

$$\frac{d^2x}{dt^2} = -g, \quad \text{subject to} \quad x(t_0) = x_0, \; \frac{dx}{dt}(t_0) = v_0$$

  to fully determine the solution to the initial value problem.

- A solution to a differential equation is called a **general solution** if it encapsulates all possible solutions to the corresponding initial-value problems. A solution is called a **particular solution** if it is associated to a specific initial value problem.

Example 3: Solve the initial value problem

$$\frac{dy}{dx} = y, \quad y(0) = 3.$$

Solution: We already know that the general solution of the differential equation is $y(x) = ke^x$ where $k \in \mathbb{R}$. It only remains to consider the initial condition $y(0) = 3$. Plugging in $x = 0$ gives us

$$y(0) = 3 = ke^{(0)} \quad \Longrightarrow \quad k = 3.$$

It follows the the particular solution we are interested in is $y(x) = 3e^x$.

Example 4: Consider a projectile thrown up into the air from the top of a cliff which is 50 meters from the ground. Suppose the projectile is subject only to the force of gravity ($F = -mg = -9.8m$ meters/second^2) and suppose the initial upward velocity of the throw is 10 m/s. Solve the initial value problem. How long does it take the projectile to reach the bottom of the cliff?

Solution: From Newton's second law, we have that $F = ma$ so that

$$-mg = m\frac{d^2x}{dt^2}.$$

With the given information, and removing the dimensions (which fortunately do match up) we can restate this as an initial value problem as

$$\frac{d^2x}{dt^2} = -9.8, \quad x(0) = 50, \; \frac{dx}{dt}(0) = 10.$$

This can be directly integrated to get

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2}\ dt = -\int 9.8\ dt = -9.8t + C.$$

We can now use the first piece of initial information to get

$$\frac{dx}{dt}(0) = 10 \quad \Longrightarrow \quad 10 = -(9.8)(0) + C \quad \Longrightarrow \quad C = 10.$$

It follows that we have

$$\frac{dx}{dt} = -9.8t + 10.$$

We can integrate this again to get

$$x(t) = \int \frac{dx}{dt}\ det = \int (-9.8t + 10)\ dt = -4.9t^2 + 10t + D.$$

The other piece of initial information gives us

$$x(0) = 50 \quad \Longrightarrow \quad 50 = -4.9(0)^2 + 10(0) + D \quad \Longrightarrow \quad D = 50.$$

It follows that the solution to the initial value problem is

$$x(t) = -4.9t^2 + 10t + 50.$$

As we might have expected, this is a parabola opening down. The vertex corresponds to the maximum height before it starts its descent to the ground. To answer the final question, we recognize that reaching the ground corresponds to setting $x = 0$. It follows that we need to find a time such that

$$-4.9t^2 + 10t + 50 = 0.$$

The quadratic formula gives the solutions $t = -2.33$ and $t = 4.37$. We can reject the negative value since it occurs before we release the projectile and conclude that the projectile will reach the ground in 4.37 seconds.

3 Existence and Uniqueness of Solutions

So far we have developed an intuition on what it means to be a solution of a differential equation, how to check if a function is in fact a solution, and how to interpret solutions geometrically. We have not, however, given any consideration to the following more basic questions:

1. Does a solution always exist? (i.e. Given an arbitrary function involving $y(x)$ and its derivatives, are we guarantee that a function $y(x)$ satisfying the equation exists?) And if it exists, does it exist over the whole domain of the independent variable (usually x in our case)?
2. If we have a solution, is it necessarily unique? (i.e. Can solutions overlap?)

In other words, we may ask questions on the existence and uniqueness of solutions. This is a significant aspect of the theoretical study of differential equations, especially for equations when the solutions are *difficult to find*. We will not spend much time on these problems, instead choosing to gain some intuition about what may happen by considering some representative examples.

The question of existence is perhaps the simplest to resolve. We will consider two examples.

Example 5: Determine a solution of

$$\left(\frac{dy}{dx}\right)^2 + y^2 = -1$$

or argue for why no solution exists.

Solution: There is no function $y(x)$ which satisfies it since the left-hand side of the expression is always positive, while the right-hand side is always negative. In this case, we do not even need to attempt to find a solution in order to know one does not exist, so the answer in this case is a definitive no, **differential equations are not guaranteed to have a solution**. (Although most of the differential equations we will consider in this course will have solutions!)

Example 6: We know that $y(x) = \tan(x)$ is a solution of $y' = 1+y^2$. We might, however, notice something strange about it: **it is not connected**! That is to say, as we are travelling along the solution, we encounter a rather abrupt jump when we hit $\pi/2$. We switch instantaneously from $+\infty$ to $-\infty$. This is not a significant concern to our mathematical analysis (everything we have done is correct!) but it might be a concern to the physical problem were are modeling. For instance, suppose we are modeling the position of some object—we cannot very well have the object explode to infinity and wrap around the other side, even though this is what the math tells us happens. In applied examples we will be careful to consider only connected (i.e.

continuous) portions of solutions, lest we run into such absurdities.

To answer the question of uniqueness, let's look again the slope fields for Examples 1 and 2. We saw at first glance that the differential equations had an infinite number of solutions, but that this was resolved by considering *initial conditions*. If we specified a particular point (x_0, y_0) in the (x, y) plane, then the initial condition through (x_0, y_0) corresponded to a *single* solution. But is this always guaranteed to be the case? Are solutions to IVPs always unique, or can different solutions overlap at a point (x_0, y_0)?

Example 7: Show that, for any $C \in \mathbb{R}$,

$$y(x) = \begin{cases} 0, & x \leq C \\ (x-C)^2, & x > C \end{cases}$$

is a solution of $\dfrac{dy}{dx} = 2\sqrt{y}$. Comment on the uniqueness of solutions.

Solution: We have that $y = 0$ implies $\dfrac{dy}{dx} = 0$ and $\sqrt{y} = 0$ trivially so $y = 0$ always satisfies $y' = 2\sqrt{y}$. To the other half of the proposed solution, we have

$$\frac{dy}{dx} = 2(x - C)$$

and

$$2\sqrt{y} = 2\sqrt{(x-C)^2} = 2|x + C| = 2(x - C)$$

where we have removed the absolute value because $x > C$ implies $x - C$, which implies $|x - C| = x - C$. It follows that both halves of the expression satisfy the differential equation, and since we have continuity and equality in derivatives at $x = C$, the function is smoothly defined at the transition. It follows that it is a solution.

We notice something a little strange when we try to consider the slope field, however (see Figure 3). The solution is a constant ($y = 0$) to the left of C and the right-half of a parabola to the right of C, but when does the transition happen? Suppose we are the point $(0, 0)$ and are travelling along the solution $y = 0$ to the right. How do we choose when we branch off to the parabola? Or even if we do? We have that $y = 0$ is always a solution, after all, so why even both considering the parabolic answer?

The problem is that solutions *overlap*. That is to say, they are not separated, as they were in the previous examples. Every solution with $C \geq 0$, for instance, goes through the point $(0, 0)$. So not only can we have solutions be non-unique due to the existence of a family of solutions, we can have them be non-unique when we restrict to solutions through a single point in the solution space as well (although this is uncommon!).

We might wonder what conditions we require of an initial-value problem in order to guarantee the existence of a *unique* solution. The answer is that we are guaranteed to have a unique solution through a point (x, y) if $\dfrac{\partial f}{\partial y}(x, y)$ is continuous at (x, y) (Theorem 1, page 24 of text). For our example, we have

$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{\sqrt{y}}$$

which is discontinuous when $y = 0$. This is why solutions were allowed to bunch at $y = 0$! (The theory behind this point will not be too important in this course, but do understand the condition for uniqueness and the interpretation of it.)

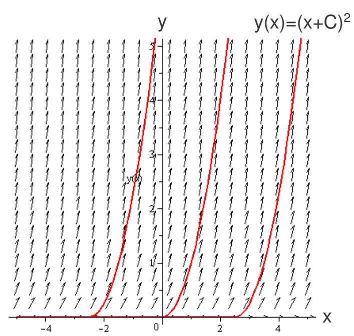

Figure 3: Slope field of $\dfrac{dy}{dx} = 2\sqrt{y}$ with $y = 0$ and the right-halves of $y_1(x) = (x + 2.5)^2$, $y_2(x) = x^2$, and $y_3 = (x - 2.5)^2$ overlain. Every solution with $C > x$ goes through the point $(x, 0)$ so that solutions intersect.

4 Separable Equations (1.4 in text)

So far we talked a great deal about solutions of differential equations—how to verify them, what properties they may have—but we have given very little thought to real big question: how to we *find* them?

Over the next few weeks, we will encounter a few specific forms which we will be able to solve by exploiting certain "tricks". The first of those, and easiest to identify, is **separable differential equations**.

To motivate this class of equations, consider the example

$$\frac{dy}{dx} = \frac{1-y}{x}.$$

Because the right-hand side depends on *both* x and y, we cannot integrate this directly with respect to x to determine the general solution. We will be need to be a little sneakier. In this example, we might notice that we can still make the problem "look like" an integration problem with respect to x by removing the y from the right-hand side and moving the differential dx to the other side. This leaves us with

$$\frac{dy}{1-y} = \frac{dx}{x}.$$

Now, not only does the right-hand side look like an integral question (with respect to x), but the left-hand side looks like an integral question as well (with respect to y). In fact, that is exactly how we will treat the equation! If we integrate (with respect to y on the left, and x on the right), we obtain

$$\int \frac{1}{1-y}\, dx = \int \frac{1}{x}\, dx \quad \Longrightarrow \quad -\ln|1-y| = \ln|x| + C \quad \Longrightarrow \quad |1-y| = \frac{e^{-C}}{|x|}.$$

Setting $k = e^{-C} > 0$, we have that $|1-y| = \dfrac{k}{|x|}$. There are a few technical details to sort out yet with the absolute value. We have the following four cases:

$$y > 1, x > 0 \quad \Longrightarrow \quad -(1-y) = \frac{k}{x} \quad \Longrightarrow \quad y = 1 + \frac{k}{x}, k > 0$$

$$y > 1, x < 0 \quad \Longrightarrow \quad -(1-y) = -\frac{k}{x} \quad \Longrightarrow \quad y = 1 - \frac{k}{x}, k > 0$$

$$y < 1, x > 0 \quad \Longrightarrow \quad (1-y) = \frac{k}{x} \quad \Longrightarrow \quad y = 1 - \frac{k}{x}, k > 0$$

$$y < 1, x < 0 \quad \Longrightarrow \quad (1-y) = -\frac{k}{x} \quad \Longrightarrow \quad y = 1 + \frac{k}{x}, k > 0$$

Recognizing that $y = 1$ (i.e. $k = 0$) is a trivial solution, we have that the sign of k does not actually matter. The general solution is $y = 1 + \dfrac{k}{x}$ for $k \in \mathbb{R}$. (This can be easily checked!)

There are a few notes worth making:

- The general trick we have performed is to separate all of the dependence on y on one side of the expression and all of the dependence on x on the other. Such differential equations are called **separable** and have the general form

$$f(y)\frac{dy}{dx} = g(x) \quad \text{or} \quad f(y)\ dy = g(x)\ dx.$$

- While everything "looks" good, we have been *very* lax in our justification of this separation (i.e. in "splitting" the differential, and integrating with respect to separate variables on the separate sides). A rigorous justification, depending on a rigorous application of the chain rule, is given in Section 1.4 of the text.

- All of the examples consider in this set of notes so far can be solved by separating the variables.

- It may seem pedantic, but it is actually important that we remember

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

  and not just $\ln(x) + C$. Solutions defined over the negative orthant will be excluded if we do not consider $|x|$. The standard tricks for handling absolute values (i.e. considering cases) will apply.

- I warned you that integration would be important for solving differential equations, and there is no class of systems that better exemplifies that than separable equations. Not only do we have to integrate to solve a separable equation, but in general we have to integrate *twice*.

Further examples are contained Section 1.4 of the textbook.