MATH 320, WEEK 3: First-Order Linear Equations, Substitution Methods, (Power) Homogeneous and Bernoulli Differential Equations

1 First-Order Linear Equations

Reconsider the first-order differential equation

$$\frac{dy}{dx} = \frac{1-y}{x}.$$

We previously showed by separating the variables that this equation has the general solution

$$y(x) = 1 + \frac{C}{x}, \quad C \in \mathbb{R}.$$

Supposing we had never heard of separation of variables, however, how might be approach the problem of solving this differential equation? Well, we might notice that we can rewrite the expression as:

$$x\frac{dy}{dx} + y = 1.$$

There is nothing in the expression dictating that we have to do this (yet!) but we can notice at least one nice thing about this form: it was easy to classify! Everything involving y and its derivatives is isolated (with respect to terms involving y), so it is a **first-order linear differential equation**. We can also see that it is easy to classify whether the expression is homogeneous or not. Since all terms involving y and its derivatives are on the left-hand side, anything appearing on the right-hand side other than zero is a sure-fire sign that the differential equation is non-homogeneous.

There is a little bit of "cheating" that has been done in rearranging the expression this way, but it is a suggestive bit of cheating. Let's consider just the left-hand side of the above expression, i.e.

$$x\frac{dy}{dx} + y.$$

If we stare this for long enough, or were born with unparalleled mathematical powers, we might notice that this can be written in a more compact form. Without justifying, for a moment, why we would *want* to do this, we might notice that this expression is the end result of the product rule for differentiation on the term xy. That is to say, we have

$$\frac{d}{dx}\left[xy\right] = x\frac{dy}{dx} + y.$$

In other words, we can take the two terms on the left-hand side and condense them into a single term, at the expense of having to recall the product rule for differentiation. At any rate, we can now rewrite the differential equation above as

$$\frac{d}{dx}\left[xy\right] = 1.$$

It should take far less mathematical insight to recognize that this is a *huge* improvement over our previous expression. The reason should be clear: we can integrate it! If we integrate the left-hand and right-hand sides by x, the Fundamental Theorem of Calculus tells us the differential on the left-hand side disappears, and the right-hand side can be evaluated as long as we know an anti-derivative of whatever the term there happens to be. That is to say, we have

$$\begin{aligned} \int \frac{d}{dx}\left[xy\right]\, dx &= \int 1\, dx \\ \implies \quad xy &= x + C, \quad C \in \mathbb{R} \end{aligned}$$

which, after dividing by x, implies that we have the general solution

$$y(x) = 1 + \frac{C}{x}, \quad C \in \mathbb{R}.$$

This is exactly the same solution we obtained before!

At this point, we should feel a little excited. We are on the path toward discovering a method for solving first-order linear differential equations. So far, the steps we took were:

1. Write with y and y' on one side,
2. Combine term on left by reversing the product rule,
3. Integrate,
4. Solve for y.

We will see in a few minutes that this is not sufficient to solve all first-order linear differential equations, but the intuition—especially the trick with the product rule—will prove to be the key to the general method.

Now consider the example

$$x\frac{dy}{dx} + 2y = 1.$$

This is only subtly different that the previous example—in fact, the only difference is the coefficient of the y term is now two. This subtle difference, however, is enough to sabotage our earlier intuition with regards to a solution method, since there is no function $f(x)$ such that

$$\frac{d}{dx}\left[f(x)\, y\right] = x\frac{dy}{dx} + 2y.$$

So what can we do?

Let's consider changing the expression (again!) but in a different way. Let's consider *multiplying* through by a single term that is a function of x. In this case, let's choose the function to be x itself. This gives us

$$x^2\frac{dy}{dx} + 2xy = x.$$

If there were any questions with regards to *why* we would want to do that, I hope they have now been answered. Using our earlier intuition with regards to the product rule, we can clearly see that we have

$$\frac{d}{dx}\left[x^2 y\right] = x^2\frac{dy}{dx} + 2xy = x.$$

Again, we can integrate to get the solution. We have

$$\int \frac{d}{dx}\left[x^2 y\right]\, dx = \int x\, dx$$

$$\implies \quad x^2 y = \frac{x^2}{2} + C, \quad C \in \mathbb{R}$$

so that the desired solution is

$$y(x) = \frac{1}{2} + \frac{C}{x^2}, \quad C \in \mathbb{R}.$$

So what was different about this example? The difference was that we had to *multiply* by some factor before we could use the product rule trick that we just discovered to get to a form we could integrate. This multiplicative factor is called an **integration factor** and is generally denoted $\mu(x)$ or $\rho(x)$. We still have to wonder how we could find integration factors. After all, how did I know to multiply by the factor x?

It is perhaps best now to scale back and consider **first-order linear systems** at their most general. In general, we have

$$\frac{dy}{dx} + p(x)y = q(x). \tag{1}$$

This is only slightly different than the forms we have been using. We now want to get all the terms involving y on the left-hand side, and also to divide through by whatever the coefficient of the derivative is so that the derivative appears by itself. Now we ask the question: *What do we have to multiply by in order to guarantee that the two terms on the left-hand side can be combined using the product rule (in reverse)*?

The answer is not obvious at first glance, but it is easy to verify. The **integration factor** we need is

$$\rho(x) = e^{\int p(x)\, dx}.$$

The details are easy to check. We know by the Fundamental Theorem of Calculus and the chain rule that $\rho'(x) = p(x)\rho(x)$ so, if we multiply the entire expression by $\rho(x)$, we have

$$\rho(x)\frac{dy}{dx} + p(x)\rho(x)y = \rho(x)q(x).$$

The left-hand side can be simplified by noting that

$$\frac{d}{dx}\left[\rho(x)y\right] = \rho(x)\frac{dy}{dx} + \frac{d\rho}{dx}y = \rho(x)\frac{dy}{dx} + p(x)\rho(x)y.$$

It follows that the differential equation can be rewritten as

$$\frac{d}{dx}\left[\rho(x)y\right] = \rho(x)q(x).$$

We can then integrate to get

$$\rho(x)y = \int \rho(x)q(x)\, dx$$

and isolate y to get the general solution

$$\begin{aligned} y(x) &= \frac{1}{\rho(x)} \int \rho(x)q(x)\, dx \\ &= e^{-\int p(x)\, dx} \int \left(e^{\int p(x)\, dx} q(x)\right)\, dx. \end{aligned}$$

That's it! So long as we can evaluate these integrals, we can solve any first-order linear differential equation.

There are a few notes worth making:

- It is not necessary to include the arbitrary constant in the integration factor integral (i.e. take $C = 0$) or the absolute value for logarithms. Both cases amount to multiplying the expression by an arbitrary constant, which does not change anything.

- On the other hand, it is very important to remember to add the constant $+C$ to the other integration (resolving the product rule).

- It is important to have the equation in the form (1). Otherwise, the given integration factor will not work.

- It is sufficient, but not recommended, to remember the general form of the solution. All of the steps in this derivation are based on tricks we know how to do, even if recognizing how to apply them might have been a little tricky.

- Whether these equations are autonomous or homogeneous depends on the forms of $p(x)$ and $q(x)$, although this particular method will work regardless.

Examples: Determine the integration factor $\rho(x)$ for the following differential equations and use it to find the general solution $y(x)$ and the particular solution for the given initial condition.

1. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x}, \quad y(1) = 1.$

2. $\dfrac{dy}{dx} + y = e^{-3x}, \quad y(0) = 2$

3. $(x+1)\dfrac{dy}{dx} - xy = e^{x}, \quad y(1) = 0.$

Solution (1): This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned}\rho(x) &= e^{\int p(x)\, dx} \\ &= e^{\int \frac{1}{x}\, dx} \\ &= e^{\ln(x)} = x.\end{aligned}$$

We can ignored the normally required $|x|$ in the $\ln(x)$ term by noticing that the two absolute value cases ($x > 0$ and $x < 0$) amount to multiplying the whole differential equation by a negative, which does not change it. Multiplying the entire expression by $\rho(x) = x$ gives us

$$x\frac{dy}{dx} + y = 1$$

which we have already seen. This was our original toy example. We already know that the general solution is

$$y(x) = 1 + \frac{C}{x}.$$

Substituting the intial value $y(1) = 1$ gives us

$$y(1) = 1 = 1 + C \quad \Longrightarrow \quad C = 0.$$

It follows that the particular solution is

$$y(x) = 1.$$

Solution (2): This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned} \rho(x) &= e^{\int p(x)\, dx} \\ &= e^{\int 1\, dx} \\ &= e^{x}. \end{aligned}$$

Multiplying the entire expression by $\rho(x) = e^x$ gives us

$$e^x\frac{dy}{dx} + e^x y = e^x \cdot e^{-3x} = e^{-2x}.$$

Recognizing that the left-hand side now must be the product rule form (expanded out), we have

$$\frac{d}{dx}\left[e^x y\right] = e^{-2x}.$$

We could jump right to this if we wanted to, but it is important to recognize the intermediate step to check that we have determined the correct integration factor. We can integrate this to get

$$\int \frac{d}{dx}\left[e^x y\right]\, dx = \int e^{-2x}\, dx$$

$$\implies \quad e^x y = -\frac{e^{-2x}}{2} + C$$
$$\implies \quad y(x) = -\frac{e^{-3x}}{2} + Ce^{-x}.$$

Using the initial condition $y(0) = 2$ gives

$$y(0) = 2 = -\frac{1}{2} + C \quad \implies \quad C = \frac{5}{2}.$$

The particular solution is therefore

$$y(x) = -\frac{e^{-3x}}{2} + \frac{5e^{-x}}{2}.$$

Solution (3): This is not in standard form, so we need to do a little work. Dividing by $(x+1)$ we arrive at

$$\frac{dy}{dx} - \frac{x}{x+1}y = \frac{e^x}{x+1}.$$

In order to determine the integrating factor, we will need to determine the integral of $-x/(x+1)$. Using the substitution $u = x+1$, we have

$$-\int \frac{x}{x+1}\,dx = \int \frac{1-u}{u}\,du = \int \left(\frac{1}{u} - 1\right)\,du = \ln(u) - u = \ln(x+1) - (x+1).$$

Recognizing that constants (i.e. the -1) do not matter for integrating factors, we arrive at

$$\rho(x) = e^{\ln(x+1)-x} = (x+1)e^{-x}.$$

Multiplying the entire expression by $\rho(x) = (x+1)e^{-x}$ gives us

$$(x+1)e^{-x}\frac{dy}{dx} - xe^{-x}y = 1.$$

It follows that we have

$$\frac{d}{dx}\left[(x+1)e^{-x}y\right] = 1$$

which can be checked. Integrating with respect to x gives

$$(x+1)e^{-x}y = x + C$$

so that the general solution is

$$y(x) = \frac{e^x}{x+1}\left(x + C\right).$$

The initial condition $y(1) = 0$ gives

$$y(1) = 0 = \frac{e}{2}(1 + C) \quad \Longrightarrow \quad C = -1.$$

It follows that the particular solution is

$$y(x) = e^x \left(\frac{x-1}{x+1} \right).$$

Other examples are available in Section 1.5 of the text.

2 Substitution Methods

Many first-order differential equations do not fall directly within the classes of separable or first-order linear systems. Nevertheless, many common identifiable classes of differential equations can be manipulated into one of these two forms via the use of a carefully selected *variable substitution*. We will look at the following examples:

1. **(Power) Homogeneous equations:** Differential equations of the general form
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$
A substitution of the form $v = \frac{y}{x}$ produces a *separable* differential equation in v and x of the form
$$x\frac{dv}{dx} = F(v) - v.$$

2. **Bernoulli equations:** Differential equations of the general form
$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$
A substitution of the form $v = y^{1-n}$ produces a *first-order linear* differential equation in v and x of the form
$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

There are a few notes worth making before we delve too deeply into these methods.

- While the given formulas are known and sufficient to solve most problems, as with first-order linear equations it is hoped that it is the *method* which is memorized, not the end formula. In other words, remember the required variable substitutions for the two types of equations.

- As with any problem involving variable substitutions involving derivatives, it is helpful to write out the tree of variable dependences. In particular, for the differential equations we are looking at, where we are looking for a function $y = y(x)$ (i.e. y as a function of x), if we define a variable transformation $v = v(x, y)$, we have the tree given in Figure 1, so that (according to the chain rule) the total derivative of v with respect to x is given by

$$\frac{dv}{dx} = \frac{\partial v}{\partial y}\frac{dy}{dx} + \frac{\partial v}{\partial x}.$$

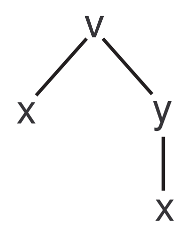

Figure 1: Variable dependence tree for $v = v(x, y)$, where $y = y(x)$.

3 (Power) Homogeneous Differential Equations

Consider the differential equation

$$2xy\frac{dy}{dx} = x^2 + y^2.$$

It should not take much arguing to convince yourself that this differential equation is neither separable nor first-order linear. We need an alternative method for such differential equations.

One possibility is to choose an appropriate variable substitution. In this case, the necessary substitution is

$$v(x, y) = \frac{y}{x}.$$

With this variable substitution, we have

$$y = xv \quad \Longrightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

so that the differential equation can be rewritten in terms of the variables v and x as

$$\begin{aligned} 2x(xv)\left(x\frac{dv}{dx} + v\right) &= x^2 + (xv)^2 \\ \Longrightarrow \quad 2x^3 v\frac{dv}{dx} &= x^2 + x^2v^2 - 2x^2v^2 \\ \Longrightarrow \quad 2x^3 v\frac{dv}{dx} &= x^2(1-v^2) \\ \Longrightarrow \quad \frac{2v}{1-v^2}\, dv &= \frac{1}{x}\, dx. \end{aligned}$$

Why this substitution helps us should now be clear. While the differential equation was not easy to solve in the variables y and x, in the variables v and x it reduces to a separable differential equation, which is among the most straight forward classification of differential equations to identify and solve. We still have some work to do, however. Continuing, we have

$$\begin{aligned} \int \frac{2v}{1-v^2}\, dv &= \int \frac{1}{x}\, dx \\ \Longrightarrow -\ln|1-v^2| &= \ln|x| + C, \quad C \in \mathbb{R} \\ \Longrightarrow |1-v^2| &= \frac{k}{|x|}, \quad k > 0. \end{aligned}$$

Now that the integration step has been resolved, we would like to return to the original variables x and y. We started with $v = y/x$, so we now have

$$\begin{aligned} \left|1 - \left(\frac{y}{x}\right)^2\right| &= \frac{k}{|x|} \\ \Longrightarrow \quad \left|x^2 - y^2\right| &= k\frac{x^2}{|x|} = k|x|, \quad k > 0. \end{aligned}$$

Again, there are technical details we need to worry about about regarding the absolute values. This situation will arise often enough that we can be a little non-rigorous in the steps which follow, but we *must* recognize that the absolute value is there in order to capture the entire class of solutions (otherwise we only get the half with $k > 0$).

We recognize that the equation breaks down into the following two cases. For the first case, we have

$$x^2 - y^2 = kx \implies y^2 = x^2 - kx \implies y = \pm\sqrt{x^2 - kx}, \quad k > 0.$$

For the second case, we have

$$-(x^2 - y^2) = kx \implies y^2 = x^2 - kx \implies y = \pm\sqrt{x^2 + kx}, \quad k > 0.$$

Once again recognizing that $k = 0$ (i.e. $y = \pm x$) is a trivial solution to the differential equation, we have the final general solution

$$y = \pm\sqrt{x^2 + Cx}, \quad C \in \mathbb{R}.$$

Wow! That was a lot of work for a relatively modest-looking answer, but nobody said solving differential equations was going to be easy. We should stop to make a few notes on this process.

- This differential equation belongs to a class of first-order differential equations called **(power) homogeneous differential equations**. A differential equation is called (power) homogeneous if it can be written in the form
$$\frac{dy}{dx} = \frac{A_1 x^{a_{11}} y^{a_{12}} + \cdots + A_n x^{a_{n1}} y^{a_{n2}}}{B_1 x^{b_{11}} y^{b_{12}} + \cdots + B_m x^{b_{m1}} y^{b_{m2}}}$$
where $a_{11} + a_{12} = \cdots = a_{n1} + a_{n2} = b_{11} + b_{12} = \cdots = b_{m1} + b_{m2}$. In other words, it is homogeneous if the sum of the powers for each term add up to the same value. This notion can be easily be generalized to terms with more than just two variables, although we will not need such generalizations for this course.

- Every (power) homogeneous differential equation can be written in the form
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$
by dividing by an appropriate power of x on the top and bottom of the previous form. The substitution $v = \dfrac{y}{x}$ is guaranteed to reduce such differential equations into a separable differential equation in v and x! (In other words, the technique we used in the example will *always* work, although, as we saw, we may still run into some tricky integration.)

- It is important to recognize differential equations which look (power) homogeneous but which in fact are not. For example, the differential equations

  $$\frac{dy}{dx} = x + y$$

  and

  $$\frac{dy}{dx} = x^2 + 2xy + y^2$$

  are *not* (power) homogeneous because there is denominator on the right-hand side with powers of x and y (the power is effectively zero, whereas the power of the numerator is two).

- I will make the distinction between homogeneous and power homogeneous differential equations. The reason for this is unfortunate: within the study of differential equations there are *two* accepted definitions of what constitutes a homogeneous differential equation, and these definitions are very different. Previously we defined homogeneous differential equations to be those with no terms which did not contain the dependent variable (y) or any of its derivatives. Usually context will dictate which meaning is implied, but just to be clear I will attempt to use *power* homogeneous to refer to the class of differential equations we were just introduced to. Notice, however, that the textbook *does not* make the distinction.

For more examples, see Section 1.6 of the text.

4 Bernoulli Differential Equations

We have seen how (power) homogeneous first-order differential equations can be transformed into separable equations by application of a fairly simple variable substitution. It turns out that there is a general class of differential equations which can be transformed into our other canonical solution class, first-order linear equations.

Consider the differential equation

$$3xy^2 \frac{dy}{dx} = 3x^4 + y^3.$$

It should not take much arguing (again) to convince ourself that this equation is not separable, is not first-order linear, and is not even homogeneous (although it is close). Based on the methods we have established so far, we are basically stuck, but we are able not going to stop there. Let's try to rearrange this equation to get it as close to the first-order linear form as possible. We have

$$\frac{dy}{dx} = x^3y^{-2} + \frac{1}{3x}y \quad \Longrightarrow \quad \frac{dy}{dx} - \frac{1}{3x}y = x^3y^{-2}.$$

We actually have not done too poorly! In fact, it is only the term on the right-hand side that presents a problem. In particular, we are not happy with the y^{-2} and would like to make it go away.

Consider the substitution $v = y^3$. We want to rewrite this differential equation in y and x as a differential equation in v and x. This will require solving for the differential and all of the y terms. We have

$$y = v^{1/3} \quad \Longrightarrow \quad \frac{dy}{dx} = \left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx}$$

and $y^{-2} = v^{-2/3}$. It follows that the differential equation can be rewritten as

$$\left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx} - \frac{1}{3x}v^{1/3} = x^3v^{-2/3}.$$

Multiplying across by $3v^{2/3}$ we arrive at

$$\frac{dv}{dx} - \frac{1}{x}v = 3x^3.$$

When we look at this, we notice that, quite remarkably, the non-linear term has disappeared. This is a linear equation in v and x! We know how to solve these types of equations. We have the integration factor

$$\rho(x) = e^{-\int \frac{1}{x}\,dx} = e^{-\ln(x)} = \frac{1}{x}.$$

This gives us

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = 3x^2 \quad \Longrightarrow \quad \frac{d}{dx}\left[\frac{1}{x}v\right] = 3x^2$$

$$\Longrightarrow \quad \frac{1}{x}v = x^3 + C \quad \Longrightarrow \quad v = x^4 + Cx.$$

We are not, of course, completely done. The original questions was a differential equation with respect to y and x, so we need to change by to our original variables. We have

$$y^3 = x^4 + Cx \quad \Longrightarrow \quad y(x) = \sqrt[3]{x^4 + Cx}.$$

This was a rather remarkable solution method, but what intuition was underlying it? It turns out that this differential equation belongs to a class of differential equations called **Bernoulli differential equations**. We pause to make the following notes about them:

- The general form of a Bernoulli differential equation is

  $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

  and the required substitution is $v = y^{1-n}$. This is guaranteed to produce a first-order linear differential equation in v and x of the form

  $$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

- Notice that there are two values which are troublesome for this transform, $n = 0$ and $n = 1$. For $n = 0$ we have $v = y$, which is trivial, and for $n = 1$ we have $v = 1$, which is meaningless. Our concern, however, turns out to be very premature. Returning to the original form of the equations, we notice that $n = 0$ and $n = 1$ both correspond to a linear first-order differential equation in the first place (for $n = 0$ this is direct, and for $n = 1$ we just have to move the term on the right to the left-hand side).

- It is worth noting that this holds for *all* values of n other than $n = 0$ and $n = 1$. That is to say, we can consider fractional powers (e.g. $n = 1/2$, $n = 7/5$, $n = 92/13$, etc.) and negative powers ($n = -3$, $n = -4/9$, $n = -103$, etc.).

For more examples, see Section 1.6 of the text.