MATH 320, WEEK 4:
Exact Differential Equations, Applications

1 Exact Differential Equations

We saw that the trick for first-order differential equations was to recognize the general property that the product rule from differentiation yields, as if by design, a form that looks like a first-order linear equation. That is to say, we have

$$\frac{d}{dx}\left[f(x)y\right] = f(x)\frac{dy}{dx} + f'(x)y.$$

This certainly *looks like* a first-order linear differential equation—all we have to do is set this equation equal to something (potentially a function of x) and we are good to go. When we investigated these problems from the other direction, trying to reverse the product rule, we recognized that we were always able to do so after (potentially) multiplying by an appropriate *integration factor*.

We might realize that there is another differentiation operator which produces a very similar form. If we consider a general function $F(x, y)$, recognizing the dependence of y of x, we have from the chain rule that

$$\frac{d}{dx}\left[F(x, y)\right] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

This certain looks like a first-order differential equation. The difference is that $F_x(x, y)$ and $F_y(x, y)$ are allowed to be functions of *both* x and y. Worst still, they are allowed to be *nonlinear* functions of y. At any rate, this forms a general class of differential equations known as **exact** differential equations. They have the general form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{1}$$

where $M(x, y) = F_x(x, y)$ and $N(x, y) = F_y(x, y)$ for some function $F(x, y)$. They are also commonly written

$$M(x, y)dx + N(x, y)dy = 0.$$

There are a few notes worth making:

- Exact differential equations are not generally linear. In other words, this is a method for solving first-order *nonlinear* differential equations.

- The general solution for an exact equation is the implicit form $F(x, y) = C$.

- Although this is a distinct class of differential equations, it will share many similarities with first-order linear differential equations. Importantly, we will discover that there is often (although not always!) an integration factor required to make a differential equation in the "exact" form. This integration factor will take a different form than that of first-order linear equations.

- The textbook does not consider integration factors for exact equations (presumably due to space concerns). That's their loss!

The question then becomes, if we have a general differential equation of the form (1), how do we know if it is exact? The answer comes to us from recognizing the equality of mixed-order partial derivatives. For a general twice differentiable function $F(x, y)$, we have

$$\frac{\partial^2}{\partial y \partial x} F(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) \quad \Longrightarrow \quad \frac{\partial}{\partial y} F_x(x, y) = \frac{\partial}{\partial x} F_y(x, y)$$
$$\Longrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

It can be shown that this is a necessary and sufficient condition for exactness (see Theorem 1 on page 70 of the text). This is an easy check, but it will not tell us how to find the general solution. For that, we consider an example.

Example 1: Show that the following differential equation is exact and use this observation to find the general solution:

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2 \right) dy = 0.$$

We have $M(x, y) = 4xy^{1/2}$ and $N(x, y) = \dfrac{x^2}{y^{1/2}} + 2$. The required condition for exactness is easy to check:

$$\frac{\partial M}{\partial y} = \frac{2x}{y^{1/2}} = \frac{\partial N}{\partial x}.$$

It follows that the equation is exact and, consequently, that there is a solution of the form $F(x, y) = C$. It remains to find the solution. How might we accomplish this?

The key is to notice that the differential equations give rise to the system of equations

$$\frac{\partial F}{\partial x} = M(x, y) = 4xy^{1/2}$$
$$\frac{\partial F}{\partial y} = N(x, y) = \frac{x^2}{y^{1/2}} + 2.$$

This can be solved by integrating either expression by the respective variable of the partial derivative. The first expression gives

$$F(x, y) = 2x^2y^{1/2} + g(y)$$

where we have to include an arbitrary function of y (i.e. the $g(y)$) because partial differentiation with respect to x would eliminate such a term. We now solve for $g(y)$ by taking the derivative of F with respect to the *other* variable, y. We have

$$\frac{\partial F}{\partial y} = \frac{x^2}{y^{1/2}} + g'(y).$$

We can see by comparing this equation with the previous system that we need to have $g'(y) = 2$. It follows that $g(y) = 2y + C$ so that the general solution is

$$2x^2y^{1/2} + 2y = C.$$

It is worth making a few notes on this process:

- It is important to remember that integrating a partial derivative requires us to add an additional term *of the other variable.*
- It is a general property that the solution will only be represented in implicit form. In other words, do not worry too much about solving for y in the final steps.

Now consider being asked to solve the differential equation

$$(4xy)dx + (x^2 + 2y^{1/2})dy = 0.$$

We notice immediately that this is just the previous example multiplied through by $y^{1/2}$. We suspect that this equation has the same solutions, and the same methods will apply, but we can see that

$$\frac{\partial M}{\partial y} = 4x \neq 2x = \frac{\partial N}{\partial x}.$$

In other words, the equation is no longer exact! This is a problem. We only know how to solve equations of this form if they are exact. We seem to be stuck.

The resolution comes by recognizing where the difference between the two equations came. We can change this expression into an exact form by dividing through by $y^{1/2}$ (or multiplying through by $y^{-1/2}$, if you prefer). It should be clear then that—just as with first-order linear equations—sometimes we will need to multiply through by some factor (also called an **integration factor**!) in order to get the equation in the form we can use.

We might wonder if *all* equations of the form (1) can be made exact by multiplication by an integration factor. This was what happened for first-order linear differential equations, so it is not an unfair question. The answer in this case, however, is unfortunately a pronounced **NO**. There are many differential equations of the form (1) which cannot be manipulated so that they are exact. The question then becomes, which differential equations can be? Are there are conditions which guarantee a differential equation of the form (1) can be made exact by multiplication by an appropriate integration factor? And, if so, what is that integration factor?

The answer to these last questions is a fortunately **YES**. We have the following conditions and associated integration factors:

Proposition 1.1. *Consider a general differential equation of the form (1). Then:*

1. *If* $R(x) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N$ *is a function of* x *alone, then the integration factor*

$$\rho(x) = e^{\int R(x)\, dx}$$

   will make (1) exact.

2. *If* $R(y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)/M$ *is a function of* y *alone, then the integration factor*

$$\rho(y) = e^{\int R(y)\, dy}$$

   will make (1) exact.

We will not justify these forms (although it is a good exercise!). Let's consider how they work for our specific example.

We need to check one or the other of the above conditions. We have

$$\frac{\partial M}{\partial y} = 4x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x.$$

To check whether the first condition is satisfied, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{4x - 2x}{x^2 + 2y^{1/2}}\right) = \left(\frac{2x}{x^2 + 2y^{1/2}}\right).$$

Since this is not a function of x alone, the first condition fails and we are not allowed to construct an integration factor out depending on x.

Now consider the second condition. We have

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{2x - 4x}{4xy}\right) = -\frac{2x}{4xy} = -\frac{1}{2y}.$$

Since this is a function of y alone, we are allow to construct an integration factor out of it. Setting $R(y) = -1/(2y)$, we have

$$\rho(y) = e^{\int R(y)\, dy} = e^{-\int \frac{1}{2y}\, dy} = e^{-\frac{1}{2}\ln(y)} = \frac{1}{y^{1/2}}.$$

This is exactly integration factor we expected! Multiplying through the expression by $\rho(x) = y^{-1/2}$ gives

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2\right) dy = 0.$$

This is the earlier expression, which we have already shown in exact, and for which we already know the solution! The only trick was determining an appropriate integration factor. It took a little more work than in the case of linear first-order differential equations, but nevertheless we were able to accomplish the task.

There are a few notes worth making:

- We may (once again) exclude constants and absolute values in the integration required to determine the form of the integration factor.
- It will be very important to keep the conditions on the variables x and y straight (though practice!). The key terms are M_y and N_x, so that the coefficient of dx has a y derivative taken, and the coefficient of dy has an x derivative taken. If the wrong derivatives are evaluated, the methods will not work.

Example: Determine the solution of

$$y\cos(x)dx + (1 - y^2)\sin(x)dy = 0.$$

Solution: We might notice that this equation is separable, but ignoring that for the time-being, we will treat as an exact (or nearly exact) equation. To check for exactness, we compute

$$M_y = \cos(x) \neq (1 - y^2)\cos(x) = N_x.$$

So that differential equation is not exact. In order to check for an integration factor, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{\cos(x) - (1 - y^2)\cos(x)}{(1 - y^2)\sin(x)}\right) = \left(\frac{y^2\cos(x)}{(1 - y^2)\sin(x)}\right).$$

This is clearly not a function of x alone, so we may remove it from consideration. The other condition gives

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{(1 - y^2)\cos(x) - \cos(x)}{y\cos(x)}\right)\left(\frac{-y^2\cos(x)}{y\cos(x)}\right) = -y.$$

Since this is a function of y alone, we set $R(y) = -y$ and evaluate the integration factor

$$\rho(y) = e^{\int R(y)\, dy} = e^{-\int y\, dy} = e^{-\frac{y^2}{2}}.$$

We now multiply the expression through by this term. We have

$$ye^{-\frac{y^2}{2}}\cos(x)dx + (1 - y^2)e^{-\frac{y^2}{2}}\sin(x)dy = 0.$$

This gives the system of necessary equations

$$\frac{\partial F}{\partial x} = M(x, y) = ye^{-\frac{y^2}{2}}\cos(x)$$
$$\frac{\partial F}{\partial y} = N(x, y) = (1 - y^2)e^{-\frac{y^2}{2}}\sin(x).$$

The obvious choice (I hope!) is to integrate the first expression with respect to x. We have

$$F(x, y) = \int \frac{\partial F}{\partial x}\, dx = ye^{-\frac{y^2}{2}}\sin(x) + g(y).$$

Taking the derivative of this with respect to y yields

$$\frac{\partial F}{\partial y} = e^{-\frac{y^2}{2}}\sin(x) - y^2e^{-\frac{y^2}{2}}\sin(x) + g'(y) = (1 - y^2)e^{\frac{y^2}{2}}\sin(x) + g'(y).$$

Comparing this with the second equation gives $g'(y) = 0$ so that $g(y) = C$. This gives the general (implicit) solution

$$F(x, y) = ye^{-\frac{y^2}{2}}\sin(x) = C.$$

2 Applications - In-Flow / Out-Flow Models

One popular application of differential equations (and in particular, first-order linear differential equations) is in modeling the amount (or concentration) of a substance in a well-stirred tank/vessel subject to constant in-flow and out-flow. Common simple applications are:

- an industrial mixing tank with an entry pipe (pumping the chemical of interest in) and an exit pipe;
- a lake with a inflow (say, a river) feeding a pollutant from upstream and an outflow (also, a river) flowing downstream;
- a tub or sink with a steady inflow (say, a faucet) and a steady outflow (say, a drain).

In all cases, the basic question is the same: If we know the in-flow, and the out-flow, can we determine what actually happens *inside* the tank/lake/tub/etc.?

To answer this question, we must translate this description from words into math. At the most basic level, we believe that

$$[\text{rate of change}] = [\text{rate in}] - [\text{rate out}].$$

That is to say, at each instance in time, we believe that the rate of change of the overall *amount* of the quantity of interest to equal the amount that is flowing in minus the amount that is flowing out. The question of characterizing the dynamics is therefore only a matter of characterizing the in-flow and the out-flow! Our knowledge of differential equations should handle the rest.

To characterize the in-flow rate, we need a few pieces of information. Firstly, we are likely to be given the overall mixture flow rate in, as well as the concentration of the quantity of interest within that in-flowing mixture. For example, we might know the amount of water which flows into a lake every day, or every week, and we might know the concentration of a particular pollutant within that volume of water. The rate of the amount of the pollutant flowing in is therefore

$$[\text{rate in}] = [\text{volume in}] \times [\text{concentration}]$$

since

$$[\text{volume in}] \times [\text{concentration in}] = \left[\frac{\text{volume}}{\text{time}}\right] \times \left[\frac{\text{amount}}{\text{volume}}\right] = \left[\frac{\text{amount}}{\text{time}}\right]$$

The out-flow is slightly different. Since we are assuming (for simplicity!) that the tank/lake/tub is well-mixed, we may assume that the concentration of the quantity of interest is the same *everywhere* in the tank/lake/tub. In particular, wherever the outflow is located, and however quickly it is removing mixture from the tank/lake/tube, we have

$$[\text{rate out}] = [\text{concentration}] \times [\text{volume out}] = \frac{[\text{amount}]}{[\text{volume}]} \times [\text{volume out}]$$

since

$$\frac{[\text{amount}]}{[\text{volume}]} \times [\text{volume out}] = \left[\frac{\text{amount}}{\text{volume}}\right] \times \left[\frac{\text{volume}}{\text{time}}\right] = \left[\frac{\text{amount}}{\text{time}}\right].$$

The key difference here is that the amount in the above derivation is the *current* amount of the quantity of interest. In other words, it is the unknown function/variable we are trying to model! Another wrinkle is that the volume is the *current* volume of the tank. If the volume of the in-flow and the volume of the out-flow do not balance, the volume of the tank may not be fixed and may in fact be a function of time (imagine filling a bathtub, or emptying a mixing tank).

Example 1: Suppose that there is a factory built upstream of Lake Mendota (volume 0.5 km^3) which introduces a new pollutant to a stream which pumps 1 km^3 of water into the lake every year. Suppose that the net outflow from the lake is also 1 km^3 per year and that the concentration of the pollutant in the inflow stream is 200 kg/km^3. Set up an initial value problem for the amount of pollutant in the lake and solve it. Assuming there is initially no pollutant in the lake, how much pollutant is there are one month? What is the limiting pollutant level?

Solution: We need to set up the model in the form [rate of change]=[rate in]−[rate out]. If we let A denote the amount of the pollutant (in kg), we have

$$[\text{rate of change}] = \frac{dA}{dt}.$$

In order to determine the rate in, we notice that the amount (in kg) coming from the inflow can be given by

$$\begin{aligned}[\text{rate in}] &= [\text{volume rate in}] \times [\text{concentration in}]\\ &= (1\ \text{km}^3/\text{year})(200\ \text{kg/km}^3) = 200\ \text{kg/year}.\end{aligned}$$

The rate out is given by

$$\begin{aligned}[\text{rate out}] &= [\text{volume rate out}] \times [\text{concentration out}] \\ &= (1 \text{ km}^3/\text{year}) \left(\frac{A}{0.5} \text{ kg/km}^3 \right) = 2 \text{ kg/year}.\end{aligned}$$

We can see the units have worked as desired. We can drop them and just focus on the initial value problem

$$\frac{dA}{dt} = 200 - 2A, \quad A(0) = A_0.$$

This is a first-order linear differential equation which in standard form is given by

$$\frac{dA}{dt} + 2A = 200.$$

We can see that we have $p(x) = 2$ and $q(x) = 200$. The necessary integration factor is

$$\rho(t) = e^{\int 2\, dt} = e^{2t}$$

so that we have

$$\begin{aligned} & e^{2t}\frac{dA}{dt} + 2e^{2t}A = 200e^{2t} \\ \implies \quad & \frac{d}{dt}\left[e^{2t}A\right] = 100e^{2t} \\ \implies \quad & e^{2t}A = 100e^{2t} + C \\ \implies \quad & A(t) = 100 + Ce^{-2t}. \end{aligned}$$

In order to solve for C, we use $A(0) = A_0$ to get

$$A(0) = A_0 = 100 + C \quad \implies \quad C = A_0 - 100.$$

This gives the solution

$$A(t) = 100 + (A_0 - 100)e^{-2t}.$$

For this form, we can easily answer the stated questions. Given an initial pollutant level of zero (i.e. $A_0 = 0$), we have

$$x(t) = 100 - 100e^{-2t}.$$

After one month has passed, we have $t = 1/12$ so that the amount of pollutant is given by

$$x(1/12) = 100 - 100e^{-2(1/12)} \approx 15.3528 \text{ kg}.$$

We can also easily determine the limiting pollutant level by evaluating

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left[100 + (A_0 - 100)e^{-2t}\right] = 100.$$

In other words, no matter what the initial amount is in the lake, we will always converge toward 100 kg of pollutant distributed throughout the lake. (This should make some sense. We imagine that the limiting level is going to be when the rate in and the rate out are balanced. That occurs for this model when $200 = 2A$ which implies $A = 100$.)

Example 2: Consider a 50 gallon tank which is initial filled with 20 gallons of brine (salt/water mixture) with a concentration of 1/4 lbs/gallon of salt. Suppose that there is an inflow tube which infuses 3 gallons of brine into the tank per minute with a concentration of 1 lbs/gallon. Suppose that there is an outflow tube which flows at a rate of 2 gallons per minute. Set up and solve a differential equation for the amount of salt in the tank. How much salt is in the tank when the tank is full?

Solution: This is slight different than the previous example because the volume of mixture in the tank *changes* because the inflow and outflow volume rates are different. There is more mixture flowing into the tank than flowing out. Nevertheless, we can incorporate this into our model by noting that the volume of the tank at time t can be given by

$$V(t) = 20 + (3 - 2)t = 20 + t.$$

We can now complete the model as before. We have

$$\frac{dA}{dt} = (3)(1) - (2)\frac{A}{20+t} = 3 - \frac{2A}{20+t}, \quad A(0) = 20(1/4) = 5.$$

Again, this is a first-order linear differential equation. We can solve it by rewriting

$$\frac{dA}{dt} + \left(\frac{2}{20+t}\right) A = 3$$

and determining the integrating factor

$$\rho(t) = e^{\int 2/(20+t)\, dt} = e^{2\ln(20+t)} = (20+t)^2.$$

This gives

$$
\begin{aligned}
& (20+t)^2\frac{dA}{dt} + 2(20+t)A = 3(20+t)^2 \\
\implies \quad & \frac{d}{dt}\left[(20+t)^2 A\right] = 3(20+t)^2 \\
\implies \quad & (20+t)^2 A = (20+t)^3 + C \\
\implies \quad & A(t) = (20+t) + \frac{C}{(20+t)^2}.
\end{aligned}
$$

Using the initial condition $A(0) = 5$, we have

$$A(0) = 5 = 20 + \frac{C}{400} \quad \implies \quad C = -6000$$

so that the particular solution is

$$A(t) = (20+t) - \frac{6000}{(20+t)^2}.$$

To answer the question of how much salt will be in the tank when the tank is full, we notice that the tank will be full when $V(t) = 20 + t = 50$, which implies $t = 30$ (i.e. it will take thirty minutes). This gives

$$A(30) = (20+30) - \frac{6000}{(20+30)^2} = 50 - \frac{6000}{2500} = 47.6.$$

It follows that there will be 47.6 lbs of salt in the tank when it is full.