MATH 320, WEEK 6: Linear Systems, Gaussian Elimination, Coefficient Matrices

We will now switch gears and focus on a branch of mathematics known as **linear algebra**. There are a few notes worth making before beginning:

- Linear algebra is a distinct branch of mathematics from differential equations—in fact, there are no derivatives at all! We will see later in the course that the applications of linear algebra to the study of differential equations are plentiful, but we can be forgiven if we do not see the connections immediately. For now, we will treat the subject matter as its own distinct entity.

- It is no coincidence that we have focused much attention on *linear* differential equations and are now transitioning into the study of *linear* algebra. The studies over the next five-to-six weeks will allow us to answer questions about general-order linear differential equations—i.e. not just *first-order* linear differential equations, which could be solved by using the trick of finding an integrating factor.

- While linear algebra is very important in the study of linear (and nonlinear!) differential equations, its applications extend far beyond that into areas such as statistics, economics, computer science, control theory—pretty well anywhere there is an equation to be solved. It is one of the foundational branches of mathematics—it simply cannot be escaped.

To motivate the study of linear algebra, consider being asked to solve the system of linear equations

$$\begin{aligned} x - y &= 4 \\ 2x + 3y &= 3. \end{aligned}$$

This is a linear system of two equations in two unknowns (x and y). By a **solution** to this system, we mean a set of values x and y which satisfy *both* equations simultaneously. How could we handle something like this?

Our first observation is that, just like with differential equations, it is exceptionally easy to *verify* whether something is a solution to given a given system of equations—all we have to do is plug the proposed solution into the given equations. For instance, suppose we have the proposed solutions (a) $(x, y) = (1, 1)$, (b) $(x, y) = (4, 0)$, and (c) $(x, y) = (3, -1)$. We can see easily that the first proposed solution is not a solution since $x = 0$, $y = 0$ does not satisfy either equation. For the second proposed solution, we can see that $(4) - (0) = 4$ so that the first equation is satisfied, but it is *not a solution* because $2(4) + 3(0) = 8 \neq 3$. Only the third proposal works, since $(3) - (-1) = 4$ *and* $2(3) + 3(-1) = 3$. In fact, this is the unique solution of the set of equations.

If we were asked to solve this without knowing the solution, we would not find ourselves particularly overwhelmed. We could simply solve for x (or y) in one of the equations, and then substitute this into the other equation. This would solve for one of the variables, and the remaining equation would allow us to solve for the other. For this example, we have

$$x - y = 4 \quad \Longrightarrow \quad x = 4 + y$$

so that

$$\begin{aligned} 2x + 3y = 3 \quad &\Longrightarrow 2(4 + y) + 3y = 3 \\ \Longrightarrow \quad 8 + 2y + 3y = 3 \quad &\Longrightarrow \quad y = -1. \end{aligned}$$

Back substitution yields

$$x = 4 + y = 4 + (-1) = 3.$$

Remark: It is worth taking a step back and considering what it means for a system of equations to be *linear*. It is analogous to what we mean by a graph being linear (a line, i.e. $y = mx + b$) or a differential equation being linear. We simply mean that the unsolved variables of interest appear in their own terms (i.e. are separated by addition or subtraction) and are not allowed to be modified by anything more complicated than a multiplicative constant. Examples of terms which are *nonlinear* are:

1. $\sqrt{x}$
2. $\sin(x)$
3. $1/x$

or even terms involving more than one unsolved variable such as xy or y^z. These terms will be called *nonlinear*, as they would have been if they involved the unknown function $y(x)$ (or any of its derivatives) during our differential equations portion of the course. Just as in the case of differential equations, nonlinear equations will be much harder to solve than linear ones.

For instance, consider being asked to solve the following nonlinear equation in a *single* variable:

$$e^{-x} = x.$$

Simply graphing the curves $y = e^{-x}$ and $y = x$ shows us that there must be a solution to the expression, but what is it? It certainly cannot be found by rearranging the expression into the form x equals to something, since no algebraic rearrangement permits us to isolate x. The best method we have (and what your computer uses) is numerical approximation using something such as Newton's method.

We will see that with linear systems such problems simply do not arise. Through our study of linear algebra, we will see that we are able to develop methods which allow us to firmly answer such questions for linear systems—and answer many of questions about the systems as well.

1 Gaussian Elimination

Now consider a more complicated system of equations. Suppose we are asked to solve the following system of three equations in three unknowns:

$$\begin{aligned} x - 2y + z &= 4 \\ -3x + y + 2z &= 13 \\ x + y - z &= -6. \end{aligned}$$

We could still (and always will be able to) follow our earlier intuition of solving for one variable at a time, and then using back substitution. That is to say, we *could* follow the following algorithm:

1. Solve for x (in terms of y and z) in the first equation.
2. Substitute this result in the second equation.
3. Solve for y (in terms of z) in the second equation.
4. Substitute this result in the third equation.
5. Solve for z.
6. Substitute z back in to solve for y.
7. Substitute z and y back in to solve for x.

This method would work, but would take a little bit of time. We would like to develop a short-hand method for going through this process which requires less writing—like synthetic division for long-division of polynomials, or the u/du, dv/v integration table for integration by parts.

Our primary observation is that when we solve for x in the first equation and substitute it in the second, what we are really doing is **eliminating** x from the second equation. This is not the only avenue available to us for accomplishing this task. We might notice that we could simply add three of the first equation to the second equation and get

$$
\begin{aligned}
3(x - 2y + z) &= 3(4) \\
+(-3x + y + 2z) &= (13) \\
- - - - - - - - - - - - \\
-5y + 5z &= 25.
\end{aligned}
$$

This equation has the same solution set as the original set—we have not changed anything. We could check that we could have obtained the second equation by solving for x directly in the first equation and substituting into the second. This is the same process!

If we perform this process of eliminate to the third equation as well, we have

$$
\begin{aligned}
(x - 2y + z) &= (4) \\
+(-1)(x + y - z) &= -(-6) \\
- - - - - - - - - - - - - - \\
-3y + 2z &= 10.
\end{aligned}
$$

Replacing the corresponding original equations with these new (equivalent) ones, we have the new system

$$
\begin{aligned}
x - 2y + z &= 4 \\
-5y + 5z &= 25 \\
-3y + 2z &= 10.
\end{aligned}
$$

We notice that the second two equations represent a system of two equations in two unknown, like we have already dealt with. We will, however, continue with our current intuition. We want to eliminate the variable y from the last equation by finding multiples of the last two equations which eliminate it. It is more complicated than the previous example, but we have

$$\begin{array}{c} 3(-5y+5z) = 3(25) \\ -5(-3y+2z) = -5(10) \\ ----------- \\ 5z = 25. \end{array}$$

This clearly implies that $z = 5$. We can substitute this back into $-3y+2z = 10$ to get $-3y + 2(5) = 10 \Longrightarrow y = 0$. We then have $x + 2(0) + (5) = 4 \Longrightarrow x = -1$. It follows that the solution is $(x, y, z) = (-1, 0, 5)$.

This certainly does not seem like less work, but consider the following observation: at each one of these steps, the basic structure of the equations has remained the same. That is to say, at each step we have three equations in three unknowns (x, y, and z). The general structure for such a system is

$$\begin{aligned} a_1x + a_2y + a_3z &= a_4 \\ b_1x + b_2y + b_3z &= b_4 \\ c_1x + c_2y + c_3z &= c_4. \end{aligned}$$

We have performed a number of operations which have changed the individual equations, but the *structure* has not changed, only the *coefficients* of x, y, and z have. If we could find a short-hand way to represent such systems without having to write x, y and z at every step, we would save ourselves a lot of writing!

To that end, we define the **coefficient matrix** M to be the grid (gridlines not shown!) with the coefficients of the linear system in the appropriate boxes, i.e.

$$M = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{array}\right].$$

The general idea behind this representation of a linear system is that we will be able to do the same intuitive steps outlined above without writing x, y, and z at every step. Let's formalize the operations we just used to find the solution $(x, y, z) = (-1, 0, 5)$ and systemize the procedure.

We are allowed to perform the following **elementary row operations** on the coefficient matrix M:

1. Interchange any two rows R_i and R_j.

2. Scale rows R_i by a non-zero constant.

3. Add two rows R_i and R_j to form a new row.

The purpose of formally stating these operations is that we can perform them on the coefficient matrix to eliminate variables from specific lines *without changing the solution set of the corresponding linear system of equations.* It is very important that we apply these operations properly and recognize how the correspond to the underlying linear system—otherwise, we will end up with the incorrect answer.

Let's apply these operations to the previous example, with the ultimate goal of eliminating x from the second and third line, and then eliminating y from the third line. We have

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -3 & 1 & 2 & 13 \\ 1 & 1 & -1 & -6 \end{array}\right] \xrightarrow{R_2'=3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 1 & 1 & -1 & -6 \end{array}\right]$$

$$\xrightarrow{R_3'=R_1-R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & -3 & 2 & 10 \end{array}\right] \xrightarrow{R_3'=3R_2-5R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & 0 & -5 & -25 \end{array}\right]$$

We recall that the last line corresponds to $(0)x + (0)y - 5(z) = -25$ so that we have $z = 5$. We could back substitute this to get our previous solution.

This process is called **Gaussian elimination** and the matrix form we have obtained is called **row echelon form**. In order to be in row echelon form, we need to have that:

1. Every row of zeroes (if applicable) appears below every line with non-zero entries.

2. Every row R_i (except for the last row) is followed by a row R_{i+1} which satisfies the following: if row R_i has zeroes in its first n columns, then row R_{i+1} has zeroes in (at least) its first $n+1$ columns.

It should be obvious that this corresponds to the structure we expect from eliminating variables. If we eliminate x from all but the first equation, the resulting coefficient matrix will have zeroes in the first column of every row except the first.

Let's continue this example a little further, though. It turns out that the back substitution steps back also be carried out by using elementary row operations by eliminate the variable z from the first two equations, and then eliminating y from the first. We have

$$\xrightarrow{R_3'=(-1/5)R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2'=R_2-5R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_2=-(1/5)R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1'=R_1+2R_2-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

This process is called **Gauss-Jordan elimination** and the matrix form we have obtained is called **row-reduced echelon form**. In order to be in row-reduced echelon form, we need to have that:

1. The matrix is in row echelon form.
2. The leading (i.e. first) coefficient in each row is a one.
3. In each column which has a leading one, all other coefficients are zero.

The advantage of having a matrix in row-reduced echelon form should be obvious. The underlying system of equations corresponding to the final matrix form is

$$\begin{aligned} (1)x + (0)y + (0)z &= -1 \\ (0)x + (1)y + (0)z &= 0 \\ (0)x + (0)y + (1)z &= 5. \end{aligned}$$

In other words, it directly gives the solution $(x, y, z) = (-1, 0, 5)$.

2 Other Possibilities

So far, the examples we have seen have had a single solution. We have not given any consideration as to whether, given an arbitrary system of linear equations, we always end up with a single solution. Reconsider the previous example, with a slight modification to the last equation:

$$\begin{aligned} x - 2y + z &= 4 \\ -3x + y + 2z &= 13 \\ x - z &= -6. \end{aligned}$$

As with the previous example, we set up our coefficient matrix and begin performing Gaussian elimination. We have:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -3 & 1 & 2 & 13 \\ 1 & 0 & -1 & -6 \end{array}\right] \xrightarrow{R_2'=3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 1 & 0 & -1 & -6 \end{array}\right]$$

$$\xrightarrow{R_3'=R_1-R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & -2 & 2 & 10 \end{array}\right] \xrightarrow{R_3'=2R_2-5R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Something appears to have gone terribly wrong. The last equation corresponds to the equation $0 = 0$, but this does not tell us anything at all meaningful at all. The remain two equations are **underdetermined**. We have three equations to solve for, but only two meaningful equations within which to do so.

To answer what is going on in this example, let's consider some geometrical points in two dimensions. A linear system of two equations in two unknowns looks like

$$\begin{aligned} a_1x + a_2y &= a_3 \\ b_1x + b_2y &= b_3. \end{aligned}$$

We can rearrange these into two equations of the familiar form $y = mx + b$. That is to say, they are *lines*, and the solution corresponds to the *intersection* of these lines. It should not take much geometrical convincing that there are three possibilities:

1. The lines intersect at a unique point, i.e. there is a single solution.
2. The lines are parallel and do not intersection, i.e. there are no solutions.
3. The lines are parallel and overlap, i.e. there are an infinite number of solutions.

Although the situations are more difficult to visualize, this intuition generalizes to an arbitrary number of dimensions! It is always the case that a linear system of equations has either no solution, one solution, or an infinite number of solutions. It is impossible for a linear system of equations to have exactly two solutions, or twelve solutions, or a thousand solutions.

To see which case we are in for our previous example, let's consider the two remaining equations. We have

$$\begin{aligned} x - 2y + z &= 4 \\ -5y + 5z &= 25. \end{aligned}$$

Let's continue to put this into row-reduced echelon form. We have

$$\xrightarrow{R_2'=-(1/5)R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1'=R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This corresponds to the system

$$\begin{aligned} x - z &= -6 \\ y - z &= -5. \end{aligned}$$

It should not be hard to convince ourselves that this system has an *infinite* number of solutions. It we choose the **parametrization** $z = t$, we have solution set given by

$$\begin{aligned} x &= -6 + t \\ y &= -5 + t \\ z &= t. \end{aligned}$$

This was obtained by subbing $z = t$ into the previous expressions and rearranging. What this equation tells us is that *any* value of t in the above expression corresponds to a solution of the system. For instance, if we select $t = 0$, we have the point $(x, y, z) = (-6, -5, 0)$, which we can easily show satisfies the original system of equations. If we pick $t = 3$, we have the point $(x, y, z) = (-3, -2, 3)$, which again can easily be seen to be a solution of the original system of equations. The really important point to recognize is that *any* t value will work, there is an infinite number of solutions. This is the general case when systems are underdetermined—i.e. when there are more variables to solve for than equations to solve with.

To see what else can happen, consider the (again) slightly modified system of equations

$$\begin{aligned} x - 2y + z &= 4 \\ -3x + y + 2z &= 13 \\ x - z &= -5. \end{aligned}$$

we set up our coefficient matrix and begin performing Gaussian elimination. We have:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -3 & 1 & 2 & 13 \\ 1 & 0 & -1 & -5 \end{array}\right] \xrightarrow{R_2' = 3R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 1 & 0 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{R_3' = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & -2 & 2 & 9 \end{array}\right] \xrightarrow{R_3' = 2R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -5 & 5 & 25 \\ 0 & 0 & 0 & 5 \end{array}\right].$$

This looks very similar to our previous example, and we might be tempted to continue on by parametrizing the first two equations. It will all be in vain, however. We recall that the last equation means

$$(0)x + (0)y + (0)z = 5.$$

In other words, the equation says $0 = 5$, which is a mathematically nonsensical statement. It does not matter what the first two equations evaluate to, there can be no solution to this system of equations because there is no solution to the last equation. We will say that the system is **inconsistent**, which means that there is no solution.

This is the third possible case. And the somewhat remarkable thing about linear systems is that is *all* that can happen. We can either have:

1. A unique solution—in which case we can find it;

2. An infinite number of solutions—in which case we can parametrize the set of solutions; or

3. No solution—in which case we say the system is inconsistent.

There are no other cases to consider, and no other methods to learn. Performing Gauss-Jordan elimination to find the row-reduced echelon form is sufficient to determine which case we are in, and if there are solutions, it will find them. All we need to do is practice!