MATH 320, WEEK 8:
Determinants

1 Determinants

We saw in previous lectures that the condition for *invertibility* of a 2×2 matrix

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$$

was $ad - bc \neq 0$. By analogy with the real numbers, this condition $ad - bc \neq 0$ is enough to make sure that we do not "divide by zero" in our matrix algebra system—and indeed, if $ad - bc = 0$, we do end up dividing by zero in our row reduction procedure for determining the inverse of a 2×2 matrix.

We should note that this is quite a bit less transparent than we might have hoped. With real numbers, it was very easy to see if we were dividing by zero—there is a zero, we see it, we identify it, and we stay away from it. For matrices, however, it might not be obvious that we are in fact "dividing by zero" since none of the entries in the matrix need be zero—even for small matrices.

It is clear, therefore, that we want to keep track of the value of the term $ad - bc$. A good start is to give it a proper name.

Definition 1.1. *The **determinant** of a 2×2 matrix is given by*

$$det(A) = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc.$$

We have the following notes:

- We can now restate our earlier intuition using the determinant. We have that a matrix is invertible if and only if $\det(A) \neq 0$.

- The applications of determinants in matrix algebra go far beyond simply determining whether a matrix is invertible or not. It is one of the fundamental parameters of a square matrix, not just invertible matrices (another is the *trace*, which we will define shortly). We will become very good at computing determinants.

Example: Find the determinant of the following matrices and use it to determine which of the matrices are invertible:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}.$$

Solution: We have

$$A = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-4)(2) = 11$$

and

$$B = \begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix} = (-1)(-6) - (3)(2) = 0.$$

It follows that A is invertible and B is not.

2   Higher-Order Determinants

We might wonder what happens for square matrices with more than two dimensions. Let's consider the question of what conditions guarantee that a general 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is invertible.

In order to compute this directly, we would need to solve for the general form of the inverse, i.e. we would need to row reduce

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array} \right].$$

This is something that (in principle) we can do, daunting as it is. If we were to perform the operations successfully, we would find that there is a single term of the form "1/stuff" which factors out of each term and thus tells us that the matrix can be inverted if and only if "stuff $\neq 0$". Naturally, of course, we call this term the **determinant** of a 3×3 matrix and have the same necessary and sufficient condition $\det(A) \neq 0$ for invertibility of A.

The trouble is that, in general, *this term is awful.* And it gets even worse! For a 3×3 matrix, the determinant involves six individual terms each involving three of the coefficients (the 2×2 had two terms in two

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coefficients). As we increase from three to four dimensions, the required term for inversion is twenty-four terms in four coefficients each. (In general, for an $n \times n$ matrix, the determinant has $n!$ terms with n coefficients.)

We cannot be expected to derive and remember a unique formula for each of these dimensions. Rather, we would like to build a *general method* by which determinants can be determinant—for any dimension. To accomplish this, we must introduce the following important concepts.

Definition 2.1. *The (i,j)-**minor** M_{ij} of the matrix general $n \times n$ matrix A is the determinant of the submatrix of A produced by removing the ith row and j^{th} column from A. The (i,j)-**cofactor** A_{ij} of A is determined by $A_{ij} = (-1)^{i+j} M_{ij}$.*

Notes:

- Since we are removing one row and one column from A, the resulting submatrix of A is an $(n-1) \times (n-1)$ matrix.

- The cofactor A_{ij} has the form

$$A_{ij} = \begin{cases} M_{ij} & \text{when } i+j \text{ is even} \\ -M_{ij} & \text{when } i+j \text{ is odd.} \end{cases}$$

- We notice that, as things stand right now, this does not help us very much since we only know how to calculate determinants for 2×2 matrices. That means we can only calculate minors and cofactors for 3×3 matrices. But there is a hint of something to come here: if we can use the minors and/or cofactors to calculate the determinants for 3×3 matrices, we are in the position where we can *inductively* extend our 2×2 determinant formula to any dimension! (i.e. Use it to determine the 3×3 case, then use that to determine the 4×4 case, etc.)

Example: Determine M_{12}, A_{12}, M_{31}, and A_{31} for

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 5 & -3 \end{bmatrix}.$$

Solution: To determine M_{12} we need to remove the first row and second column of A. We have that

$$M_{12} = \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = (-1)(-3) - (1)(1) = 2.$$

Since we have that $i + j = 3$ is odd, we have that $A_{12} = (-1)M_{12} = -2$. To determine M_{31} we need to remove the third row and the first column of A. We have that

$$M_{31} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (1)(1) - (-2)(0) = 1.$$

Since we have that $i + j = 4$ is even, we have that $A_{31} = M_{12} = 1$.

We are now (finally!) prepared to define a determinant for a square matrix A of arbitrary dimension.

Definition 2.2. *The **determinant** of an $n \times n$ matrix A is given by the formula*

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ji} A_{ji}$$

where $i = 1, \ldots, n$ is fixed (and arbitrary).

This formula is a little daunting at first glance (didn't we start this whole discussion by saying we wanted to avoid unnecessary complication?) but the basic idea is this:

1. Arbitrarily fix a row or a column (this is the fixed index i).

2. For each component in the row (usually starting at the left or the top), compute the product of the entry and the corresponding cofactor (i.e. the determinant of the matrix with that row and column *removed*).

3. Add the results across each entry in the chosen row or column.

4. That's it!

Example: Determine the determinant of the matrix A given by

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 5 & -3 \end{bmatrix}.$$

Solution: Our first step is to choose a row or column to keep fixed. Without any guidance or tricks, we might as well just choose the first row

(corresponding to the entries 3, 1, and -2). In order to compute the determinant, we must compute

$$\det(A) = (3) \begin{vmatrix} 0 & 1 \\ 5 & -3 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= (3) \left[(0)(-3) - (5)(1) \right] - (1) \left[(-1)(-3) - (1)(1) \right] + (-2) \left[(-1)(5) - (0)(1) \right]$$

$$= 3(-5) - (1)(2) + (-2)(-5)$$

$$= -7$$

That wasn't so bad! It was a little bit of work, for sure, but it was significantly easier than trying to invert the general form of the matrix and remember the six term constant which would fall out. And yet it can be shown (with a little *more* work) that this is the term you would get factoring out in the denominator after performing elementary row operations to determine A^{-1}.

We should stop now to make a few notes about this process:

- Since this formula allows us to compute the determinant of $(n+1) \times (n+1)$ matrices based on knowledge of the determinants of $n \times n$ matrices, knowing the formula for determinants of 2×2 matrices allows us to compute the determinants of *all* dimensions. (It should be noted that, beyond 3 or 4 dimensions, this is still a lot of work! In general, for big matrices this is a computer exercise.)

- We must be careful to remember how the *signs* of the cofactors are determined. They alternate along the rows and columns. In general, they follow the pattern

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

- It is a property of determinants that it does not matter which row or column along which we expand the determinant. We will always get the same answer. This can allow us to take shortcuts in our computation. In particular, any row or column with zeroes is going to allow us to simplify our work. For instance, we chose the second column in

the previous example, we would have

$$\det(A) = -(1) \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} + (0) \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} - (5) \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}$$
$$= -(1)\left[(-1)(-3) - (1)(1)\right] - (5)\left[(3)(1) - (-2)(-1)\right]$$
$$= -(1)(2) - (5)(1)$$
$$= -7.$$

Let's stop now to state a few properties of determinants:

1. A general $n \times n$ matrix is invertible if and only if $\det(A) \neq 0$. That is to say, the determinant is truly the property which will tell us whether we are "dividing by zero" when we perform the matrix algebra equivalent of division.

2. Consider an $n \times n$ matrix A and its transpose A^T. Then we have

$$\det(A) = \det(A^T).$$

3. Consider an $n \times n$ matrix A which is invertible. Then we have

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

4. If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

5. If any two rows or columns of A are scalar multiples of one another, then $\det(A) = 0$.

6. If A and B are two $n \times n$ matrices which differ only by two interchanged rows or interchanged columns, then $\det(B) = -\det(A)$.

7. If A and B are two $n \times n$ matrices which differ only in that a single row (or column) of B is a scalar multiple of the corresponding row (or column) in A, then if k is the scalar constant we have $\det(B) = k\det(A)$.

3 Applications of Determinants

Determinants can be used explicitly to solve linear systems of equations $A\vec{x} = \vec{b}$. That is to say, not only can they be used to determine whether A is invertible so that we can write $\vec{x} = A^{-1}\vec{b}$, but they can be used to explicitly solve for the values $x_i, i = 1, \ldots, n$.

For details, see pages 212-214 in the text.