MATH 320, WEEK 9:
Linear Combinations, Linear Independence/Dependence

1   Linear Independence/Dependence

Let's reconsider the process underlying Gaussian elimination in a little more depth.

We saw a linear system in n equations and n unknowns could be written as

$$A\vec{x} = \vec{b}$$

where $\vec{x}$ and $\vec{b}$ were n-dimensional vectors and A was an $n \times n$ matrix. We realized that if there was a matrix A^{-1} such that $A^{-1}A = I$ (i.e. if A was **invertible**) then we could write the expression as

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$$

and therefore solve for the vector $\vec{x}$ using matrix operations rather than Gaussian elimination.

We then took a step back and asked the very important question of *when* a matrix A is invertible. The computation involved in finding a general inverse is cumbersome, but we were able to sidestep most of it by consideration of **determinants**. We were able to convince ourselves that a matrix A was invertible if and only if $\det(A) \neq 0$. And so we only have to compute the determinant in order to determine whether this algebraic process of solving linear systems of n equations in n unknowns would succeed.

Now we will consider the process from a different direction. We will need to take a step backward in order to take a step forward. Rather than considering matrix operations, we will reconsider the process underlying Gaussian elimination.

Suppose we want to solve the following linear system of three equations in three unknowns

$$\begin{aligned} 3x_1 - x_2 + 4x_3 &= 0 \\ x_1 - x_2 \quad\quad\ &= 0 \\ x_1 - 2x_2 - 2x_3 &= 0. \end{aligned}$$

This type of linear system is called **homogeneous** because there are no terms which are independent of the variables of interest (x_1, x_2, and x_3). Homogeneous systems are easy to identify because they can be put into the form $A\vec{x} = \vec{0}$ where $\vec{0}$ is the vector of all zeroes (or the appropriate dimension).

If we perform Gaussian elimination, we find that (after omitting a few steps!)

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

By now, of course, we are well practiced in using this form to determine the solution set. We have that there is no leading one corresponding to x_3, so we set it equal to an arbitrary parameter $x_3 = t$. Solving the other equations gives $x_1 = -2t$ and $x_2 = -2t$ so that in vector notation we have that the solution set is given by

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = t \left[\begin{array}{c} -2 \\ -2 \\ 1 \end{array}\right].$$

Any value of t produces a set of values for x_1, x_2, and x_3 which satisfies the original system of equations. We also notice that this can be written in the condensed vector form $\vec{x} = t\vec{v}$ where $\vec{v} = [-2 \ -2 \ 1]^T$.

As an algorithm, this is a nice and tidy way to resolve the issue of variables which are not assigned to leading ones—we can apply it to *any* linear system in exactly the same way and always arrive at a vector form solution set. But it should be unsatisfying in some way since we have not given too much consider to what is *actually happening* in the Gaussian elimination process to lead to this result (until now!). In particular, we might want to consider the questions:

1. What does it mean for a row reduced matrix to have a row of zeroes? (And how can such a situation arise?)

2. How do we interpret a solution set that depends on *vectors*?

It turns out that the answer to both of these question will depend on a mathematical construct known as **vector spaces** (but we are not quite there yet).

To address the first question of how a row of zeroes can arise—the question which will be the focus of most of this week—consider the following reasoning:

1. The elementary row operations which led us to a row of zeroes applied to the *rows* of A. So let's consider the rows of the matrix A individually as vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. In other words, let's write

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & -1 & 0 \\ 1 & -2 & -2 \end{bmatrix} = A = \left[\begin{array}{ccc} 3 & -1 & 4 \\ \hline 1 & -1 & 0 \\ \hline 1 & -2 & -2 \end{array} \right] = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

where $\vec{a}_1 = [3 \ -1 \ 4]$, $\vec{a}_2 = [1 \ -1 \ 0]$, and $\vec{a}_3 = [1 \ -2 \ -2]$.

2. The only operations we were allowed to perform on the rows were

    (a) switching their order (not interesting!);

    (b) scaling by some nonzero factor (e.g. things like $2\vec{a}_1$ or $-3\vec{a}_2$, etc.); and

    (c) adding rows (e.g. $\vec{a}_1 + \vec{a}_2$, $\vec{a}_2 + \vec{a}_3$, etc.).

    If we keep track of all the elementary row operations we perform in the Gaussian elimination process, it should not take much justification to convince ourselves that *every new row we encounter* can be written in the form
    $$\vec{a}_{new} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3 \tag{1}$$
    for some real values $c_1, c_2, c_3 \in \mathbb{R}$. That is to say, every new row $\vec{a}_{new}$ in the elimination process can be obtained by some combination or scaling and addition of the original rows $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. A sum of the form (1) is called a **linear combination** of vectors.

3. The conditions for encountering a row of zeroes in the Gaussian elimination process is now staring us in the face. We can only obtain a row of zeroes if there are $c_1, c_2, c_3 \in \mathbb{R}$ not all zero such that

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3 = \vec{0} \tag{2}$$

    (i.e. the new row, somewhere in the process of Gaussian elimination, is the zero row). If there is a set of constants c_1, c_2, c_3 not all equal to zero such that (2) is satisfied, we will say the vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are **linearly dependent**. (Otherwise, they are **linearly independent**.)

To verify tha the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly dependent for this example, we can explicitly find the values of c_1, c_2, and c_3 which satisfy (2). It

can be easily seen that what we need to solve is the linear system

$$
\begin{bmatrix} 3 & 1 & 1 \\ -1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

This can be row reduced to give

$$
\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 \\ 4 & 0 & -2 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].
$$

The solution set can be parametrized by $c_3 = t$ so that $c_1 = \frac{1}{2}t$ and $c_2 = -\frac{5}{2}t$. The interpretation of these values is a little obscure, but we can make it concrete by picking a convenient choice of t. In this case, $t = 2$ yields

$$
\{c_1 = 1, c_2 = -5, c_3 = 2\}.
$$

In other words, we have that

$$
(1)\vec{a}_1 + (-5)\vec{a}_2 + (2)\vec{a}_3 = \vec{a}_1 - 5\vec{a}_2 + 2\vec{a}_3 = \vec{0}.
$$

This can be easily verified for the rows $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ given earlier! (Although it should be pointed out that this was only one possible combination of the rows which would work. In general, we could have picked any value of $t \in \mathbb{R}$ and would have obtained a similar relationship between the three vectors.)

This leads to the following questions:

1. How can we determine if a set of vectors is linearly dependent?

2. What is the intuition behind a set of vectors being linear indepen-dent/dependent? (i.e. Why should we care?)

To answer the first question, recall that finding the required constants c_1, c_2, and c_3 for demonstrating linear dependence was equivalent to solving

$$
\begin{bmatrix} \vec{a}_1 & | & \vec{a}_2 & | & \vec{a}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

If this system has a unique solution, then that solution is $c_1 = c_2 = c_3 = 0$. Since solving a system uniquely can only be done if the matrix is invertible, and that the matrix is invertible if and only if its determinant is non-zero, we

have that the condition for linear independent of a set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$
is

$$\det \left[\ \vec{a}_1 \ \middle| \ \vec{a}_2 \ \middle| \ \vec{a}_3 \ \right] \neq 0.$$

Conversely, a set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly dependent if and only if

$$\det \left[\ \vec{a}_1 \ \middle| \ \vec{a}_2 \ \middle| \ \vec{a}_3 \ \right] = 0.$$

For our previous example, we have

$$\begin{vmatrix} 3 & 1 & 1 \\ -1 & -1 & -2 \\ 4 & 0 & -2 \end{vmatrix} = (4) \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} + (-2) \begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix}$$
$$= (4)\left[(1)(-2) - (-1)(1)\right] - (2)\left[(3)(-1) - (-1)(1)\right]$$
$$= -4 + 4 = 0.$$

So we could have determined, without solving for the c_1, c_2, and c_3 explicitly
that the set of vectors was linearly dependent! This intuition holds for any
set of n vectors with n components.

To investigate the second question (i.e. why we *care* whether a set of
vectors is linearly dependent or not), we observe that we could have written
the condition for linear dependence in a different way. We could have written

$$\vec{a}_1 - 5\vec{a}_2 + 2\vec{a}_3 = \vec{0} \quad \Longrightarrow \quad \vec{a}_3 = -\frac{1}{2}\vec{a}_1 + \frac{5}{2}\vec{a}_2.$$

This does not look that meaningful at first glance, but it actually tells us a
great deal about what it means to have a row of zeroes arise in our Gaussian
elimination process. It means that all of the information contained in the
third row of A (i.e. the third equation in our system) could be stated in terms
of information contained in the first two rows (i.e. the first two equations in
our system). In fact, it can be easily verified that the linear system of *two*
equations

$$3x_1 \ - x_2 + 4x_3 = 0$$
$$x_1 \ - x_2 \qquad = 0$$

has exactly the same solution set as the original system. Given the first two
equations, the third is completely redundant. (In fact, we could perform
this argument using any of the three rows. Given the first and third equa-
tions, adding the second adds no new information; and given the second and
third equations, adding the first adds no new information. Any two will do.)

Example: Show that the set of vectors $\{\vec{v}_1, \vec{v}_2\} = \{(1, -2), (-1, 2)\}$ is linearly dependent.

Solution The set of vectors is linearly dependent if

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = 0$$

for c_1, c_2 not all zero. This is equivalent to the system

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Our determinant test for linear independence gives

$$\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} = (1)(2) - (-1)(-2) = 0$$

which verifies that the vectors are linearly dependent.

Graphically, for vectors in $\mathbb{R}^2$, it is easy to identify whether they are linearly dependent or not. Vectors which are linear dependent must lie on the same line or, stated another way, they must be multiples or one another. (We will see, however, that this intuition does not always extend to vectors in higher dimensions. While it is true that two vectors which are multiples of one another will be linearly dependent, there are vector sets which are simply *co-planar* which have the property of being linearly dependent. For instance, the earlier set $\{(3, -1, 4), (1, -1, 0), (1, -2, -2)\}$ is linearly dependent but none of the vectors is a multiple of another.)

2 Linear Combinations and Span

In the previous section, we encountered linear combinations of two sets of vectors which were used to describe some aspect of a problem. We found that any solution $\vec{x}$ to $A\vec{x} = \vec{b}$ could be given in vector form as

$$\vec{x} = t\vec{v}$$

and any new row in our Gaussian elimination process had the vector form

$$\vec{a}_{new} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3.$$

It is clear that linear combinations of vectors play a key role in matrix algebra! It should not be surprising that a significant branch of the general

study of linear algebra devotes itself to considering the properties of linear combinations of vectors. We will pause now to briefly introduce just a few of these important notions (which will form the basis of **vector spaces**).

We start with a few notes on vector notation. We will interchangeable denote vectors as $\vec{v} = [v_1 \ v_2 \ \cdots \ v_n]$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$, depending on the context. It is often more convenient to write vectors with the commas; however, when performing matrix operations, it will be important to think of them as a subclass of matrices. Where there is ambiguity, we will assume vectors are in the orientation (row or column) suitable for matrix multiplication. We will also allow the individual vectors $\vec{v}$ to have arbitrary dimension (i.e. $\vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$), although you may think of them as two- or three-dimensional if you prefer—the intuition offered from these dimensions will apply to larger dimensions!

Throughout the following definitions, we will consider the general (finite) set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ where m is some arbitrary (finite) integer:

- A term of the form $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ where $c_1, \ldots, c_m \in \mathbb{R}$ (i.e. scaling and summation of the vectors) is called a **linear combination** of the vector set S.

- The vector set S is called **linearly dependent** if

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_m = \vec{0}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$ not all equal to zero. Conversely, if

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_m = \vec{0}$$

can only be satisfied for $c_1 = \cdots = c_m = 0$ (all coefficients are zero) then we will say the set is **linearly independent**.

- The **span** of S is defined as

$$\text{span}\,(S) = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \text{ for some } c_1, \ldots, c_m \in \mathbb{R}\}\,.$$

In other words, the span of a set of vectors is the set of all vectors which can be *reached* by a linear combination of the vectors in the set.

Example 1: Show that $(3, 1)$ is in the span of the vectors

$$S = \{(-1, -1), (2, 3), (2, -1)\}\,.$$

Solution: In order to be in the span of S, there needs to be constants c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This is equivalent to the matrix equation

$$\begin{bmatrix} -1 & 2 & 2 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

We perform Gaussian elimination to get

$$\left[\begin{array}{ccc|c} -1 & 2 & 2 & -3 \\ -1 & 3 & -1 & 1 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & -7 \\ 0 & 1 & -3 & -2 \end{array} \right].$$

It follows that the system has a solution for any $c_1 = -7 + 8t$, $c_2 = -2 + 3t$, and $c_3 = t$ where $t \in \mathbb{R}$ is an arbitrary parameter. For instance, picking $t = 0$ gives $(c_1, c_2, c_3) = (-7, -2, 0)$, corresponding to the linear combination

$$(-7) \cdot (-1, -1) + (-2) \cdot (2, 3) + (0) \cdot (2, -1) = (3, 1)$$

which can be easily verified. It should be noted that this is not the only combination which will lead to the point $(3, 1)$. If we picked $t = 1$ in the above solution set, we would have obtained $(c_1, c_2, c_3) = (1, 1, 1)$ corresponding to the linear combination

$$(1) \cdot (-1, -1) + (1) \cdot (2, 3) + (1) \cdot (2, -1) = (3, 1)$$

which again can be easily verified. For an illustration of the linear combination corresponding to $(c_1, c_2, c_3) = (1, 1, 1)$, see Figure 1.

Example 2: Show that $(1, 0, 1)$ is not in the span of

$$S = \{(2, 5, -1), (-1, -8, 3), (1, -3, -2)\}.$$

Solution: We need to show that the system

$$c_1 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -8 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
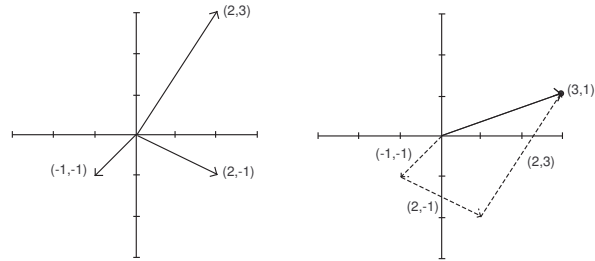
8

Figure 1: The point $(3, 1)$ can be reached as a linear combination of the vectors $(-1, -1)$, $(2, 3)$, and $(2, -1)$ and is therefore in the span of the vectors. The combination of vectors represented in this case corresponds to $c_1 = c_2 = c_3 = 1$; however, other combinations which reach $(3, 1)$ exist.

cannot be satisfies for any $c_1, c_2, c_3 \in \mathbb{R}$. The Gaussian elimination gives

$$
\left[\begin{array}{ccc|c}
2 & -1 & 1 & 1 \\
5 & -8 & -3 & 0 \\
-1 & 3 & 2 & 1
\end{array} \right]
\longrightarrow
\left[\begin{array}{ccc|c}
1 & -3 & -2 & -1 \\
0 & 5 & 5 & 3 \\
0 & 7 & 7 & 5
\end{array} \right]
$$

$$
\longrightarrow
\left[\begin{array}{ccc|c}
1 & -3 & 2 & 1 \\
0 & 5 & 5 & 3 \\
0 & 0 & 0 & -4
\end{array} \right].
$$

The last equation implies that the system is inconsistent, and consequently $(1, 0, 1)$ is not in the span of S.