MATH 320, WEEK 10:
Vector Spaces, Dimension, and Basis

1 Vector Spaces

It turns out that the concepts of linear combinations, linear independence / dependence, and span do not only arise from performing Gaussian elimination. They allow us to define and describe **vector spaces**, one of the foundational building blocks of the discipline of linear algebra. We will not fully explore these concepts, but we will make use of some of the most importation results.

Definition 1.1. *Consider a set of vectors V. We will say that V is a **vector space** if:*

1. For any $c \in \mathbb{R}$ and any $\vec{v} \in V$, we have $c\vec{v} \in V$.

2. For any $\vec{v}, \vec{w} \in V$, we have $\vec{v} + \vec{w} \in V$.

This definition seems simple enough at first glance, but we shall see that there are many subtleties which can arise. We start by making a few notes about how vector spaces relate to our interests in this course:

- The most commonly used vector spaces (and the only ones we will be focusing on in this class) are $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$ (commonly called R-two and R-three). Vectors in $\mathbb{R}^2$ take the form $\vec{v} = (v_1, v_2)$; that is to say, they have two entries and the entries are allowed to be chosen from the real numbers. In general, we can define the vector space $V = \mathbb{R}^n$ (i.e. R-n) to be the vector space with entries $\vec{v} = (v_1, v_2, \ldots, v_n)$. It is clear that we have:

  1. For any $c \in \mathbb{R}$ and any $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $c\vec{v} = (cv_1, \ldots, cv_n) \in \mathbb{R}^n$; and
  2. For any $\vec{v}, \vec{w} \in \mathbb{R}^n$, we have $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \in \mathbb{R}^n$.

- In full generality (which mathematicians love!), vector spaces need not have anything to do with conventional *vectors* at all! For instance, the class of continuous functions on the real number line forms a vector space since

1. For any $c \in \mathbb{R}$ and any continuous function $f(x)$ on $x \in \mathbb{R}$, we have $cf(x)$ is a continuous function on $x \in \mathbb{R}$; and

2. If we have two functions $f(x)$ and $g(x)$ which are continuous on $x \in \mathbb{R}$, then $f(x) + g(x)$ is a continuous function on $x \in \mathbb{R}$.

We will not deal with vector spaces more general than $\mathbb{R}^n$ in this course, but it is important to realize that the machinery we will develop can be applied to more general constructs.

We have a pretty good intuitive grasp of what $\mathbb{R}^2$ and $\mathbb{R}^3$ look like. They are the two-dimensional and three-dimensional (hyper)planes, respectively (usually indexed and graphed as the (x, y)-plane and (x, y, z)-hyperplane, respectively). We might wonder, therefore, what there is left to do in these spaces.

To consider this question, we introduce the following related concept:

Definition 1.2. *Consider a collection of vectors which for a vector space V. We will say that W is a **subspace** of V if*

1. For any $\vec{v} \in W$ we have $\vec{v} \in V$ (i.e. $W \subseteq V$);

2. For any $c \in \mathbb{R}$ and any $\vec{v} \in W$, we have $c\vec{v} \in W$; and

3. For any $\vec{v}, \vec{w} \in W$, we have $\vec{v} + \vec{w} \in W$.

It follows froms points 2. and 3. that any subspace W of V is a vector space in and of itself. Condition 1. just implies that every vector in W must also be contained within V. By convention, in this course I will usually let $V = \mathbb{R}^n$ for some n and any space $W \subseteq V$ will be considered a *subspace* of the vector space $V = \mathbb{R}^n$. We also notice that, for any vector space V, it follows that $W = V$ and $W = \emptyset$ are subspaces of V.

Consider the following examples.

Example 1: Show that $W = \left\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\right\}$ is a subspace of $V = \mathbb{R}^2$.

Solution: We have the following the following way:

1. It follows trivially that $(x, y) \in W$ implies $(x, y) \in V = \mathbb{R}^2$.

2. Consider $c \in \mathbb{R}$ and $(x, y) \in W$. This implies that we have $c(x, y) = (cx, cy)$. We want to check whether $(cx, cy) \in W$. In order to satisfy

the requirements for entry in W we need to show that $(cx)+2(cy) = 0$. In fact, we can see that

$$(cx) + 2(cy) = c(x + 2y) = 0$$

since $x + 2y = 0$ (because $(x, y) \in W$). It follows that $c(x, y) \in W$ as well!

3. Consider $(x_1, y_1) \in W$ and $(x_2, y_2) \in W$. This implies that

$$x_1 + 2y_1 = 0$$
$$x_2 + 2y_2 = 0.$$

We want to show that $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in W$, i.e. we want to show that $(x_1 + x_2) + 2(y_1 + y_2) = 0$. We have

$$(x_1 + x_2) + 2(y_1 + y_2) = (x_1 + 2y_1) + (x_2 + 2y_2) = 0$$

because $x_1 + 2y_1 = 0$ and $x_2 + 2y_2 = 0$. It follows that $(x_1, y_1) + (x_2, y_2) \in W$!

Since the points chosen were arbitrarily chosen, we have that W is a subspace of $V = \mathbb{R}^2$.

Example: Show that $W^* = \left\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\right\}$ is not a subspace of $V = \mathbb{R}^2$.

Solution: It is easy to see we have $(x, y) \in W$ implies $(x, y) \in \mathbb{R}^2$, so that condition 1. is satisfies. For the second requirement 2., however, we can see that $c \in \mathbb{R}$ and $(x, y) \in W^*$ gives the point $c(x, y) = (cx, cy)$. In order for $(cx, cy) \in W$, we need to show that $(cx) + 2(cy) = 2$ (by definition). We can see that $x + 2y = 2$ implies that we have

$$(cx) + 2(cy) = c(x + 2y) = 2c.$$

This is *not* equal to one unless $c = 1$! This is enough to ruin the argument. Since there is a value of c and a point $(x, y) \in W$ (for example $c = 2$ and $(x, y) = (2, 0)$) for which $c(x, y) \notin W$ (e.g. since $c(x, y) = 2(2, 0) = (4, 0)$ implies $x + 2y = (4) + 2(0) = 4 \neq 2$), it follows that W cannot be a subspace of $\mathbb{R}^2$.

To consider what has happened in these two examples, consider Figure 1. We can see that both W and W^* lie within $V \in \mathbb{R}^2$ as expected and that

both correspond to lines. So how are we supposed to differentiate between subspaces (like W) and sets which are not subspace (like W^*)? The answer is actually fairly simple. In order to satisfy $c\vec{v} \in W$, we must be able to move around in our space W as far along straight lines as possible; however, the line *must* be centered at $(0,0)$! If it is not, we will pass through the set and never meet it again. In fact, it can be easily verified graphically that the subspace W can be represented in vector form as

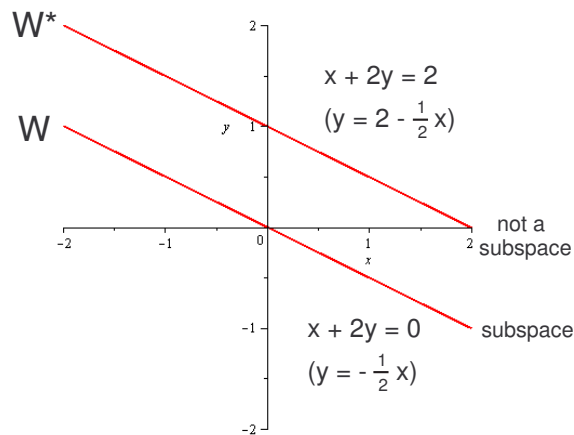$$W = \{t(2,-1), t \in \mathbb{R}\} = \text{span}\left\{(2,-1)\right\}.$$

Figure 1: Graphical representation of the set W and the set W^* embedded within $V \in \mathbb{R}^2$. Although both sets represent lines, only W passes through $(0,0)$ and so that it is a subspace of $\mathbb{R}^2$.

2 Spanning Sets, Basis and Dimension

The last examples imply that we have perhaps been a little over-zealous in our axiomatic definition of vector spaces for $\mathbb{R}^2$ (and in general, $\mathbb{R}^n$). What we really need in order to define a vector space is a set of vectors which can *reach* everywhere in the subspace W through the operations of scalar multiplication and addition.

We already have the concepts required for determining how far a set of vectors can *reach*: it is the span! Taking the span of a set of vectors

amounts to taking all possible linear combinations of the vectors so we can find *all* vectors which can be reached through these two operations. It turns out that this intuition operations in both ways, which is formalized in the following result.

Theorem 2.1. *For any set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq V = \mathbb{R}^n$, we have that $W = span(S)$ is a subspace of $\mathbb{R}^n$. Furthermore, every subspace W of $V = \mathbb{R}^n$ can be expressed as $W = span(S)$ for some (finite) set $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$.*

When the $span(S) = W$, we will say that the set S **spans** W. It is worth noting, however, that the equality must be strict. Every vector in $span(S)$ must be contained in W, and every vector in W must be in $span(S)$.

We have already seen several examples of subspaces which are generated from vectors in exactly this way. At any rate, now we can be formal about it. We have that the set

$$\text{span}\,\{(-2, -2, 1)\} = \{t(-2, -2, 1), t \in \mathbb{R}\},$$

which describes the solution set of the original linear system, is a subspace of $V = \mathbb{R}^3$. We also have that the set

$$\begin{aligned} &\text{span}\,\{(3, -1, 4), (1, -1, 0), (1, -2, -2)\} \\ &= \{c_1(3, -1, 4) + c_2(1, -1, 0) + c_3(1, -2, -2), c_1, c_2, c_3 \in \mathbb{R}\}, \end{aligned}$$

which describes the set of possible row equivalent rows of A is a subspace of $V = \mathbb{R}^3$. In other words, we have already talked about vector spaces without realizing it.

This is a tremendous step forward in describing subspaces W of $V = \mathbb{R}^n$ (and general vector spaces, as well). It says that we can describe these spaces—which in general contain an infinite number of vectors—by considering a *finite* set of generating vectors. In particular, it says that that every vector $\vec{v}$ in a subspace W of $V \in \mathbb{R}^n$ can be represented as

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

for some set $\{\vec{v}_1, \ldots, \vec{v}_m\} \subset \mathbb{R}^m$ and constants $c_1, \ldots, c_m \in \mathbb{R}$. This is pretty sweet, but there are still a few questions we might wonder about:

1. Is the set of generating vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ *unique*?

2. If they are not unique, is there some *optimal* set of such vectors? (And optimal in what sense?)

To investigate the first question, let's reconsider an earlier example. We found that the span of the set $S = \{(3, -1, 4), (1, -1, 0), (1, -2, -2)\}$ was a subspace of $\mathbb{R}^3$ (i.e. the set $W = \text{span}(S)$). It is a little redundant to say that any point in W can be reached by a linear combination of the vectors in S—that is the definition of the space!—but we can still ask the question of whether this is the *only* vector representation of the set. We will argue that it is not. Consider the following reasoning:

- Earlier we showed that the set $S = \{(3, -1, 4), (1, -1, 0), (1, -2, -2)\}$ was linearly dependent by showing that

$$(3, -1, 4) - 5(1, -1, 0) + 2(1, -2, -2) = (0, 0, 0)$$

  where clearly the coefficients of the matrices are not all equal to zero (in fact, none are).

- This condition can be rewritten as (among other things)

$$(3, -1, 4) = 5(1, -1, 0) - 2(1, -2, -2). \tag{1}$$

- What this says is that any representation of a vector in W which involves the last vector $\vec{v}_1 = (3, -1, 4)$ can be rewritten in terms of the other two vectors $\vec{v}_2 = (1, -1, 0)$ and $\vec{v}_3 = (1, -2, -2)$ by using the equation (1). It follows that we have

  $$\text{span}\,\{(3, -1, 4), (1, -1, 0), (1, -2, -2)\} = \text{span}\,\{(1, -1, 0), (1, -2, -2)\}.$$

  In other words, including the $\vec{v}_1$ in our generating set does not contribute any information about W which cannot already be found by considering just $\vec{v}_2$ and $\vec{v}_3$!

At this point, we should have some intuition about where this is going. We would not only like to represent vector spaces by a finite set of generator vectors, but we would also like this set to contain a *minimal* number of elements. In other words, we would like to describe and transverse vector spaces by using as little information as possible!

We have also developed some intuition on how we will reduce the number, if indeed we can. If a set of vectors is linearly dependent, one of the vectors (at least) can be written as a linear combination of the others. We can remove such a vector from the set and be assured that the set which remains spans the same space. Repeating this process must eventually lead to a set of vectors which is linearly independent (since a singleton set is necessarily

linearly independent). If we agree that generating sets with fewer elements are "better" than those with a greater number of elements, we have an answer to our second question.

The following definition captures this intuition.

Definition 2.1. *A set of vector $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is said to be a **basis** of a vector space V (or subspace $W \subset V$) if $span(S) = V$ and the set of vectors in S is linearly independent.*

Example: Show that $B_1 = \{(1, -1, 0), (1, -2, -2)\}$ is a basis of the earlier defined W.

Solution: We know that $span(S) = W$ from before, so it only remains to show that the vectors in S are linearly independent. This means we need to show that

$$c_1(1, -1, 0) + c_2(1, -2, -2) = (0, 0, 0)$$

has only the solution $c_1 = c_2 = 0$. We can write this as the linear system

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Gaussian elimination quickly gives

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & -2 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

It follows that $c_1 = c_2 = 0$ is the only solution and consequently that the set of vectors in S is linearly independent. It follows that S is a basis of W.

There is another way to shown linear independence of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\} \subset \mathbb{R}^n$ where $m < n$ (i.e. when the number of vectors is fewer than the number of coordinates). We have the following result.

Theorem 2.2. *The set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\} \subset \mathbb{R}^n$ where $m < n$ is linearly independent if and only if the matrix*

$$A = \left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{array} \right]$$

has some $m \times m$ submatrix for which the determinant is nonzero. Conversely, the set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is linearly dependent if every $m \times m$ submatrix of A has a zero determinant.

For the previous example, we have that

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \\ 0 & -2 \end{bmatrix}$$

and we can easily check that

$$\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = (1)(-2) - (-1)(1) = -1 \neq 0$$

so that the vectors are linearly independent by the theorem.

That is pretty sweet, but we might still wonder about how special this set *really* is. In particular, we might notice that the process we applied to reduce the number of vectors in our spanning set was not unique to $\vec{v}_3$. We could have removed $\vec{v}_3 = (1, -2, -2)$ by noticing that

$$(3, -1, 4) - 5(1, -1, 0) + 2(1, -2, -2) = (0, 0, 0)$$

$$\implies \quad (1, -2, -2) = -\frac{1}{2}(3, -1, 4) - \frac{5}{2}(1, -1, 0).$$

It follows that the set $B_2 = \{(3, -1, 4), (1, -1, 0)\}$ also spans W. In fact, it can easily be shown that this set is also linearly independent so that B_2 is *also* as basis for W. In other words, basis in general *are not unique*.

It might be disappointing to realize that vector spaces do not have *unique* bases, but we do at least notice one element in common between the sets B_1 and B_2—they have the same *number* of elements. In fact, we can fairly easily convince ourselves that this must be the case. This is formalized by the following result (and definition).

Theorem 2.3. *Suppose V is a vector space with bases B_1 and B_2. Then the number of elements in B_1 and B_2 is the same. This number if called the **dimension** of V and is denoted* dim(V).

This result allows us to relax a little bit. We may not be able to find a unique basis, but at the very least we know that the number of elements needs to minimally generate a vector space is always the same.

Let's build on this intuition but further exploring some properties and examples of bases:

- The vector spaces $V = \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$, are often assigned the **standard basis** $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is the vector with one in the ith component and zeroes elsewhere. For example, in $\mathbb{R}^3$ we have the standard basis $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, and $\vec{e}_3 = (0, 0, 1)$.

- A key feature of having a basis $B = \{\vec{v}_1, \ldots, \vec{v}_m\}$ for a vector space V is that each element $\vec{v} \in V$ can be written as a linear combination of elements in the basis, i.e.

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$$

in *exactly one way*. That is to say, once the basis has been chosen, the representation of each vector $\vec{v} \in V$ in terms of that basis is fixed. In this way, the coefficients c_i, $i = 1, \ldots, m$, can be thought of as the **coordinates** of the vector $\vec{v}$ in the coordinate system implied by the basis B.

- The concepts of linear independence, spanning sets, and bases may seem abstract in their strict mathematical formulation, but we encounter the intuition almost every day in our day-to-day lives—especially if we regularly use *maps*. When we consider the relationship between two points on a map, we often make the assumptions we have just made for bases in the mathematical realm without even realizing it. For instance, we assume:

  1. The topographical map can be described by a finite set of generating directions—in this case, the east/west (longitudinal) and north/south (latitudinal) axes. We do not need to define the set of points on the map (the space) individually or according to some abstract axioms.

  2. Each point can be defined uniquely with respect to this basis (degrees of latitude and longitude). We do not need to worry about two coordinates corresponding to the same point, or some points in the map not being assigned a coordinate (not being able to "reach" them).

  3. This basis is minimal with respect to describing the points on the map. That is to say, if we tried to add a new component (say, a northwest/southeast axis) to describe the relationship of one point to another, we would find ourselves introducing redundant information.

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4. We need two components to the basis (longitude and latitude) to describe this space because the map describes an essential two-dimensional space (ignoring curvature of the Earth). If we were to add another dimension of interest—say, to describe the position of an airplane or a satellite—we could have to add another coordinate to describe the height of the object as well. The idea of adding another coordinate to describe a direction previously not considered is exactly the concept of linear independence.

Example: Show that the set $B = \{(2, 0, -1), (1, 1, 0), (0, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

Solution: We know that $\dim(\mathbb{R}^3) = 3$ so that if the set B is linearly independent it must span $\mathbb{R}^3$. To check linear independence, we check the determinant

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (2) \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - (1) \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= (2)(1) - (1) = 1 \neq 0.$$

It follows that B is linearly independent and therefore is a basis of $\mathbb{R}^3$.

3 Matrix-Related Vector Spaces

We motivated this whole discussion on the relatively abstract topic of vector spaces on various conditions underlying matrix operations, and in particular Gaussian elimination. Now we will return to consideration of matrices.

We now introduce the following matrix-related vector spaces.

Definition 3.1. *Consider an $m \times n$ matrix A. We define:*

1. *the **row space** of A (denoted $\text{row}(A)$) to be the span of the row vectors of A;*

2. *the **column space** of A (denoted $\text{col}(A)$) to be the span of the column vectors of A; and*

3. *the **null space** of A (denoted $\text{null}(A)$) to be the set of vectors $\vec{v}$ such that $A\vec{v} = \vec{0}$.*

For example, consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

We have very easily that $\text{row}(A) = \text{span}\{(2, -3, 1), (-1, 1, 0)\}$ and that $\text{col}(A) = \text{span}\{(2, -1), (-3, 1), (1, 0)\}$. The only space requiring any work is determining $\text{null}(A)$. To do this we need to solve $A\vec{v} = \vec{0}$ which gives the linear system

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right].$$

Set $v_3 = t$, we have $v_1 = t, v_2 = t$. It follows that $(v_1, v_2, v_3) = t(1, 1, 1) = \text{span}\{(1, 1, 1)\}$. It follows that $\text{null}(A) = \text{span}\{(1, 1, 1)\}$.

There are a few notes worth making:

- These definitions apply to matrices of arbitrary (finite) dimension. We do not need to restrict our attention to square matrices!

- We will need to be care about the dimensions of the vectors in these spaces. For an $m \times n$ matrix, the vectors in the column space will be m dimensional, while the vectors in the row space and null space will be n dimensional.

- The null space of matrix also goes by the name of the *kernel* of a matrix. For the purposes of this course, I will always refer to the null space; however, be aware that other courses which deal with similar subject matter may refer to the space by this different name.

We know that defining a vector space by the span of an arbitrary set of vectors is not, in general, the best way to describe the space. In particular, we may have linear dependence within the set of vectors, so that the space can actually be generated by a smaller set of vectors. So we might wonder what the *best way* to represent these spaces is. In other words, can we easily find a *basis* for these spaces?

The answer comes to us from consideration of the row-reduced echelon form of the matrix.

Theorem 3.1. *Consider the row-reduced echelon form of an $m \times n$ matrix A. Then:*

1. *A basis for* row(A) *is given by the rows of the* row-reduced echelon matrix *which have leading ones;*

2. *A basis for* col(A) *is given by the columns of the* original matrix *which have leadings ones in the row-reduced echelon form; and*

3. *A basis for* null(A) *is given by the span of the vectors found by placing the solution* $A\vec{v} = \vec{0}$ *in vector form.*

For the example matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

we had the row-reduced echelon matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

It follows that the set $\{(1, 0, -1), (0, 1, -1)\}$ is a basis of the row(A), the set $\{(2, -1), (-3, 1)\}$ is a basis of col(A) (since the third column is not assigned a leading one in the row reduction), and $\{(1, 1, 1)\}$ is a basis of null(A) (as before).

We have the following notes to make.

- At this point, we should be growing slightly tired of hearing about the row-reduced echelon matrix and all the things we can accomplish with it. (Is there any process in matrix algebra which does not depend on it in some way? Probably not many.) At any rate, we should be realized that there is not much more arithmetic to do. The only new thing we are doing here is interpreting the end result of the elimination process in a novel way.

- It is important to recognize that the basis for the row space is given by the *actual rows* in the row-reduced matrix corresponding to leading ones. The basis for the column space is given by the columns in the *original matrix* which correspond to the leading ones. It is important to recognize this!

So far we have accomplished two things: defined the spaces, and given a minimal description of them (by determining the bases). We now want to say something about how these three spaces relate to one another. To accomplish this, firstly we will need to introduce the following definitions.

Definition 3.2. *Consider an $m \times n$ matrix A with row space row(A) and null space null(A). Then the **rank** of A is given by rank$(A) = dim(row(A))$ and the **nullity** of A is given by nullity$(A) = dim(null(A))$.*

For our previous example, we have two elements in the basis for row(A) and a single element in the basis for null(A) so that rank$(A) = 2$ and nullity$(A) = 1$.

We have the following results regarding how the row, column, and null spaces relate.

Theorem 3.2. *Consider an $m \times n$ matrix A with row space row(A), column space col(A), and null space null(A). Then we have*

1. *rank$(A) = dim(row(A)) = dim(col(A))$;*

2. *rank$(A) + nullity(A) = n$; and*

3. *If B_{row} is a basis of row(A) and B_{null} is a basis of null(A) then $B = \{B_{row}, B_{null}\}$ is a basis of $\mathbb{R}^n$.*

We can easily verify this for the previous example. We have that $dim(row(A)) = dim(col(A)) = 2$ so that the first point is satisfied, rank(A)+nullity$(A) = 2 + 1 = 3 = n$ so that the second point is satisfied, and it can be easily checked that the set

$$B = \{(1, 0, -1), (0, 1, -1), (1, 1, 1)\}$$

is a basis of $\mathbb{R}^3$ since

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$
$$= (1) \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}$$
$$= (2) + (1) = 3 \neq 0.$$

Example: Determine a basis for the row space, column space, and null space of

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & -2 \\ 1 & -1 & 3 & 4 & -3 \\ 1 & -1 & -2 & -2 & 1 \end{bmatrix}.$$

Determine the rank and nullity of A and verify that the rank-nullity theorem is satisfied.

Solution: The matrix A can be row reduced to give

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & -2 \\ 1 & -1 & 3 & 4 & -3 \\ 1 & -1 & -2 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the row space is given by the rows of the row reduced matrix with leading ones, so that we have

$$\text{row}(A) = \text{ span} \left\{(1, -1, 0, 0, -1), (0, 0, 1, 0, -2), (0, 0, 0, 1, 1)\right\}.$$

A basis for the column space is given by the columns of *the original* matrix corresponding to leading ones in the row reduced echelon form, so that we have

$$\text{col}(A) = \text{ span} \left\{(1, 1, 1, 1), (0, 1, 3, -2), (1, 1, 4, -2)\right\}.$$

A basis for the nullspace can be given by the vector form of the solution of the homogeneous system

$$A\vec{v} = \vec{0}$$

so that we have

$$\text{null}(A) = \text{ span} \left\{(1, 1, 0, 0, 0), (1, 0, 2, -1, 1)\right\}.$$

We have that $\text{rank}(A) = \dim(\text{row}(A)) = 3$ and $\text{nullity}(A) = \dim(\text{null}(A)) = 2$. It follows that

$$\text{rank}(A) + \text{nullity}(A) = (3) + (2) = 5$$

which corresponds to the number of *columns* of A. This is exactly what we expect from the rank-nullity theorem.

3.1 Change of Basis (Time Permitting)

If we accept that vector spaces in general have multiple bases and that, in general, we do not have a preference for one basis over another, we might wonder how easy it is switch from one basis to another. That is to say, suppose that a point (or vector) in our vector space V has coordinates $c_1, \ldots, c_m$ with respect to a basis B. Can we determine (easily) what the point's coordinates $c_1^*, \ldots, c_2^*$ are with respect to another basis B^*?

Consider the following two examples.

Example 1: Suppose we want to transform points from the basis $B = \{(2, 0, -1), (1, 1, 0), (0, 1, 1)\}$ to the standard basis $B^* = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Suppose we know the coordinates of a point c_1, c_2, and c_3 with respect to B. Determine the coordinates of the point with respect to B^*.

Solution: Consider this reasoning: since both B and B^* span $\mathbb{R}^3$, we can write either *basis* in terms of the other. Determining this relationship should be our first step! From then we simply need to determine how the coefficients of an individual point change when one basis is substituted for another.

In order to determine how the coefficients change, we consider the task of writing an arbitrary element $\vec{b}_i \in B$ as a linear combination of elements in $B^* = \left\{ \vec{b}_1^*, \ldots, \vec{b}_m^* \right\}$. We have the general form

$$\vec{b}_i = c_{i1}\vec{b}_1^* + c_{i2}\vec{b}_2^* + \cdots + c_{im}\vec{b}_m^*$$

for $i = 1, \ldots m$. This can be written in condensed matrix form as

$$B^*C = B$$

where

$$B^* = \left[\; \vec{b}_1^* \mid \vec{b}_2^* \mid \cdots \mid \vec{b}_m^* \; \right], \quad C = \begin{bmatrix} c_{11} & \cdots & c_{m1} \\ \vdots & \ddots & \vdots \\ c_{1m} & \cdots & c_{mm} \end{bmatrix},$$

$$\text{and} \quad B = \left[\; \vec{b}_1 \mid \vec{b}_2 \mid \cdots \mid \vec{b}_m \; \right].$$

For our specific example, we have

$$B^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Solving for c_{11}, c_{12}, and c_{13}, for instance, tell us how $(2, 0, -1)$ is indexed according to the standard basis. Of course, it is obvious that we have

$$(2, 0, -1) = (2)(1, 0, 0) + (0)(0, 1, 0) + (-1)(0, 0, 1)$$

so we must have $c_{11} = 2$, $c_{12} = 0$, and $c_{13} = -1$. In fact, the entire matrix equation is equivalent to the row form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

which is already solved! (This is the same type of system we have for solving inverses. In general, we will have to solve this, but because we chose the standard basis this was already done for us.) We have that

$$C = \left[\begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{array}\right].$$

We now want to describe a particular point with respect to the change of basis implied by C. We recall that an individual point $\vec{v}$ can be written with respect to the old basis as

$$\vec{v} = B\vec{c}$$

where $\vec{c} = (c_1, c_2, c_3)$ is the (unique) set of coordinates with respect to the basis B. In order to determine the set of coordinates with respect to the standard basis B^*, we compute

$$\vec{v} = B\vec{c} = B^*C\vec{c} = B^*\vec{c}^*$$

where $\vec{c}^* = C\vec{c}$. In other words, we can take the old coordinates with respect to B, multiply them through the matrix C we just derived, and we will obtain the new coordinates in the standard basis B^*. We have

$$\left[\begin{array}{c} c_1^* \\ c_2^* \\ c_3^* \end{array}\right] = \left[\begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] = \left[\begin{array}{c} 2c_1 + c_2 \\ c_2 + c_3 \\ -c_1 + c_3 \end{array}\right]$$

so that a point with coordinates (c_1, c_2, c_3) with respect to the old basis has coordinates $(c_1^*, c_2^*, c_3^*) = (2c_1 + c_2, c_2 + c_3, -c_1 + c_3)$ with respect to the standard basis.

To take an example, let's consider the point with coordinates $(c_1, c_2, c_3) = (1, 0, -1)$ with respect to the basis B. This gives

$$(1)(2, 0, -1) + (0)(1, 1, 0) + (-1)(0, 1, 1) = (2, -1, -2).$$

Our formula gives us the coordinates $(c_1^*, c_2^*, c_3^*) = (2c_1 + c_2, c_2 + c_3, -c_1 + c_3) = (2, -1, -2)$. This corresponds to the point

$$(2)(1, 0, 0) + (-1)(0, 1, 0) + (-2)(0, 0, 1) = (2, -1, 2)$$

as expected.

Example 2: Let's consider going the other direction. That is to say, suppose we know the coordinates of a point/vector $\vec{v}$ (c_1, c_2, c_3) with respect to the standard basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and want to determine the coordinates with respect to the basis $B^* = \{(2, 0, -1), (1, 1, 0), (0, 1, 1)\}$.

Solution: We have the same set up. We need to rewrite the vectors in the basis B in terms of the basis B^*. This gives the matrix equation

$$B^* C = B$$

as before. In this case, however, we have the coefficient matrix to solve

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array} \right].$$

In order to solve for the entries of C (i.e. to find how the first basis depends on the second) we row reduce this matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{array} \right].$$

It follows that the matrix we need is

$$C = (B^*)^{-1} = \left[\begin{array}{ccc} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 2 \end{array} \right].$$

This is great! We now know that any point/vector $\vec{v}$ with coordinates $\vec{c} = (c_1, c_2, c_3)$ with respect to B can be written

$$\vec{v} = B\vec{c} = B^* C \vec{c} = B^* \vec{c}^*$$

where $\vec{c}^* = C\vec{c}$. We have $(c_1^*, c_2^*, c_3^*) = (c_1 - c_2 + c_3, -c_1 + 2c_2 - 2c_3, c_1 - c_2 + 2c_3)$. For instance, if we are interested in the point $\vec{v} = (1, 0, -1)$ which has coordinates $(c_1, c_2, c_3) = (1, 0, -1)$ with respect to the standard basis B, we have that the coordinates are $(c_1^*, c_2^*, c_3^*) = (0, 1, -1)$ with respect to B^*. In fact, we can easily check that

$$(0)(2, 0, -1) + (1)(1, 1, 0) + (-1)(0, 1, 1) = (1, 0, -1).$$