MATH 320, WEEK 11:
Eigenvalues and Eigenvectors

1 Eigenvalues and Eigenvectors

We have learned about several vector spaces which naturally arise from matrix operations. In particular, we have learned about the *row space*, the *column space*, and the *null space*.

All of these spaces have a wide variety of uses within linear algebra and related applications (like differential equations!) but there is one very important set of vectors (and related vector space) which we have not investigated yet: the *eigenvectors* and *eigenspace*. In a sense, we have saved the best for last—there is hardly any application of linear algebra which is not influenced by these objects.

To motivate what it is we are looking for, let's first formally introduce the concept of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Definition 1.1. *Consider an $n \times n$ matrix A. The **linear transformation** associated with A is the mapping which takes vectors $\vec{v} \in \mathbb{R}^n$ to vectors $\vec{w} \in \mathbb{R}^n$ according to the relationship*

$$\vec{w} = A\vec{v}.$$

In other words, a linear transformation takes vectors (or points) in some dimension and associates them to another point in the same space. For instance, if we have a two-dimensional space (i.e. $\mathbb{R}^2$), the linear transformation $\vec{w} = A\vec{v}$ takes points in the plane to other points in the plane. We saw this on Assignment #6 with the rotation and projection matrices.

In general, linear transformations take vectors to new vectors—that is to say, they move things. We can easily check that, for example, $\vec{v} = (1, 0)$ is mapped to $\vec{w} = A\vec{v} = (-1, -3)$ and $\vec{v} = (0, 1)$ is mapped to $\vec{w} = A\vec{v} = (2, 4)$. There does not appear to be any significant pattern to *how* things move. But consider computing $\vec{w} = A\vec{v}$ with $\vec{v} = (1, 1)$ for the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}.$$

We can easily compute that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In other words, the transformation did not change anything! This is certainly not a *general* trend. So why did $\vec{v} = (1,1)$ stay put?

To more fully resolve the question we should be asking, let's consider another vector. This time, we will use $\vec{v} = (2,3)$. For this vector, we have

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

This is not exactly like the previous vector—we have not obtained the *exact same* vector we started with—but we probably recognize fairly quick that we have $\vec{w} = (4,6) = 2(2,3) = 2\vec{v}$. In other words, the transformation has *scaled* this vector by a factor of two, but the new vector still lies on the same line.

So, embedded in the linear transformation $\vec{w} = A\vec{v}$ are a pair of *invariant* directions. That is to say, there are a pair of directions where, if we start along a line extending in a particular directly, the transformation may stretch us out (or compress us), but we are never allowed to move off of the line. Let's formalize what these invariant quantities are.

Definition 1.2. *Suppose an $n \times n$ matrix A. Then we will say that λ is an **eigenvalue** of A with corresponding **eigenvector** $\vec{v}$ if*

$$A\vec{v} = \lambda\vec{v}. \tag{1}$$

This is exactly what we need! The eigenvector is the direction where the transformation $\vec{w} = A\vec{v}$ is invariant, and the eigenvalues λ represents the scaling in the invariant direction. With this new terminology added to our linear algebra lexicon, the next questions are obvious:

1. Do $n \times n$ matrices always have eigenvalues and eigenvectors?

2. If so, how can we find them?

To answer these questions, let's consider the equation (1) in more depth. What do we need to have happen in order for $A\vec{v} = \lambda\vec{v}$? For starters, we can rewrite the equation in the following way:

$$A\vec{v} - \lambda\vec{v} = \vec{0} \implies A\vec{v} - \lambda I\vec{v} = \vec{0} \implies (A - \lambda I)\vec{v} = \vec{0}.$$

This may not look like much, but it is actually a significant improvement. This is a null space problem, and we know how to perform null space problems.

We are still not done, however. At this point both λ and $\vec{v}$ have not been determined, and we know they must be carefully selected since most vectors do not satisfy the invariance principle we are looking for. What we are going to do is notice the following:

1. In order for a matrix to have a non-trivial null space, we must have nullity$(A) > 0$.

2. By the Rank-Nullity Theorem for an $n \times n$ matrix, we have rank(A)+nullity$(A) = n$. It follows that rank$(A) < n$.

3. If a square matrix does not have full rank, it follows that it is not invertible, i.e. $\det(A) = 0$.

In other words, the *only possible way* to satisfy $(A - \lambda I)\vec{v} = \vec{0}$ is to have $\det(A - \lambda I) = 0$. This is great news! We know how to compute determinants for arbitrary square matrices. If we can solve the resulting equation for λ we will have obtained the *eigenvalues* of the matrix!

This suggests the following algorithm for determining eigenvalues and eigenvectors:

1. Compute the $\det(A - \lambda I)$. This gives an equation in λ which is called the *characteristic equation*.

2. Solve $\det(A - \lambda I) = 0$ for λ. This gives the eigenvalues. (In general, there are n of them, labelled $\lambda_1, \lambda_2, \ldots, \lambda_n$.)

3. Determine the null space of $(A - \lambda I)$ for all values of λ found in part 2. This gives the eigenvectors $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$.

4. Check the equation $A\vec{v}_i = \lambda_i \vec{v}_i$ for all the pairs found! This is the easy part, but it is a good way to validate your work.

Let's reconsider the previous example. We have

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}.$$

This gives

$$A - \lambda I = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 - \lambda & 2 \\ -3 & 4 - \lambda \end{bmatrix}.$$

We can quickly compute

$$\det(A - \lambda I) = (-1 - \lambda)(4 - \lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0.$$

We can see that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, as expected. These are the scalings of the invariant vectors. We now want to compute the eigenvectors.

To $\lambda_1 = 1$, we have the equation

$$(A - \lambda_1 I) = (A - I) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}.$$

We want the nullspace, so we compute

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -3 & 3 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

The second component has no leading one, so we set $v_2 = t$ and solve $v_1 = t$, $t \in \mathbb{R}$. Taking $t = 1$ it follows that we have $\vec{v}_1 = (1, 1)$ (we can pick any vector in the span of the solution).

To $\lambda_1 = 2$, we have the equation

$$(A - \lambda_1 I) = (A - 2I) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}.$$

We want the nullspace, so we compute

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ -3 & 2 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

The second component has no leading one, so we set $v_2 = t$ and solve $v_1 = (2/3)t$, $t \in \mathbb{R}$. Taking $t = 3$ it follows that we have $\vec{v}_1 = (2, 3)$. That's it! We that have the eigenvalue/eigenvector pairs for this matrix are

$$\lambda_1 = 1, \ \vec{v}_1 = (1, 1), \quad \text{and} \quad \lambda_2 = 2, \ \vec{v}_2 = (2, 3).$$

2   The Characteristic Equation

To consider what else can happen when computing eigenvalues and eigenvectors, let's consider a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can quickly see that $\det(A - \lambda I) = 0$ gives

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc$$
$$= \lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Noticing that $a + d$ is the trace of A and $ad + bc$ is the determinant of A, we have the quadratic equation

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

$$\implies \quad \lambda = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}.$$

There are three essentially different cases which can occur with this equation:

1. **Two distinct eigenvalues**: If $\text{tr}(A)^2 > 4\det(A)$ then there will be two distinct eigenvalues.

2. **One repeated eigenvalues**: If $\text{tr}(A)^2 = 4\det(A)$ then there will be one repeated eigenvalue.

3. **Complex conjugate pair eigenvalues**: If $\text{tr}(A)^2 < 4\det(A)$ then there will be a complex conjugate pair of eigenvalues.

There are a few notes worth making about this:

- These results generalize to arbitrary dimensions! For a general $n \times n$ matrix A, the characteristic equation is always an nth order polynomial in λ. It is a general property of polynomials (often called the Fundamental Theorem of Algebra) that their roots are either real numbers (possibly repeated) or complex conjugate pairs (possibly repeated). That is to say, there are no new cases which pop out as we bump up to considering 3×3 matrices, or 4×4, etc.

- Writing the characteristic equation for a 2×2 matrix in terms of the trace and determinant is *not* required, but it is useful. If we compute the trace and determinant first, we now only have two things to keep track of in subsequent computations rather than the original four (a, b, c, and d).

- Since eigenvectors correspond to invariant *spaces* (i.e. they are a null space, and hence a vector space), we may take any vector in the space to describe the space. That means we can take any value of the arbitrary parameter $t \in \mathbb{R}$ that we like. We will usually choose it to be the smallest value for which allows all of the terms in $\vec{v}$ are integers.

- How we handle the eigenvectors in each of these three cases turns out to be a little bit different. This will be handled on a case-by-case basis. If the eigenvalues are complex, we will have complex eigenvectors. If eigenvalues are repeated, it is possible (although not always the case) that we will not end up with a full set of n eigenvectors.

- In general practice, eigenvalues and eigenvectors are *messy*! The examples in this class will be carefully manufactured to work out well.

2.1 Complex Conjugate Eigenvalues

Let's consider another example. Let's try to find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}.$$

We perform the analysis exactly as before. We have

$$\begin{vmatrix} 3 - \lambda & 2 \\ -1 & 1 - \lambda \end{vmatrix} = (3 - \lambda)(1 - \lambda) + 2$$
$$= \lambda^2 - 4\lambda + 5 = 0.$$

This has no obvious roots, so we plug this into the quadratic formula to get

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm \frac{\sqrt{-4}}{2} = 2 \pm \sqrt{-1} = 2 \pm i.$$

We have not see the imaginary number $i = \sqrt{-1}$ yet in this course, but we will not be able to escape it now! It is a somewhat surprising fact of the real world that *imaginary* numbers pop up frequently and tell us meaningful information about the *real* quantities they are modeling. We will see this when we return to consideration of differential equations. Many processes which involve *oscillations* give rise to complex values.

For the purposes of this course, we will need to know only a few things about the imaginary number i:

1. Any complex number $z \in \mathbb{C}$ can be written as the sum of a real part $Re(z)$ and an imaginary part $Im(z)$ in the following form:

$$z = Re(z) + Im(z) \cdot i.$$

2. In order to add two complex numbers $z, w \in \mathbb{C}$ we just add the real and imaginary parts separately:

$$z + w = (Re(z) + Re(w)) + (Im(z) + Im(w)) \cdot i.$$

3. When multiplying complex numbers, we will need to remember that $i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$. Notice that this is a real number! This means that when we multiply complex numbers, we will have recollect the real and imaginary part after resolving all of the multiplications. The imaginary parts will not necessarily stay imaginary. For example, if $z = 1 + i$ and $w = 2 - i$, we have

$$z \cdot w = (1 + i)(2 - i) = 2 + i - i \cdot i = 2 + i - (-1) = 3 + i.$$

4. Later on in the course, we will need the famous *Euler's formula*

$$e^{ix} = \cos(x) + i \sin(x).$$

If you have not see this formula before, you are probably doing a double-take. How can an exponential, two basic trigonometric functions, and the imaginary number i be related? Nevertheless, this identity can be (relatively) easily verified by taking the Taylor series expansion of the left-hand and right-hand sides. We will not need to understand this formula, but we will need to remember it.

Complex analysis is a very rich area of mathematical analysis—and very useful in many areas of applied mathematics, including dynamical systems, cosmological modeling, and fluid mechanics—but these operations are the extent to which we will touch on it.

Now let's find the eigenvectors. We use the same equation as before. We want to find the null space of $A - \lambda_1 I$ where $\lambda_1 = 2 + i$ (we will ignore the other conjugate $\lambda_2 = 2 - i$ for the time being). We have

$$A - \lambda_1 I = \begin{bmatrix} 3 - (2 + i) & 2 \\ -1 & 1 - (2 + i) \end{bmatrix} = \begin{bmatrix} 1 - i & 2 \\ -1 & -1 - i \end{bmatrix}.$$

To find the nullspace, we compute

$$
\begin{bmatrix} 1-i & 2 & \bigm| & 0 \\ -1 & -1-i & \bigm| & 0 \end{bmatrix}
$$

$$
\overset{(1+i)R_1}{\Longrightarrow} \begin{bmatrix} (1-i)(1+i) & 2(1+i) & \bigm| & 0 \\ -1 & -1-i & \bigm| & 0 \end{bmatrix}
$$

$$
\Longrightarrow \begin{bmatrix} 2 & 2+2i & \bigm| & 0 \\ -1 & -1-i & \bigm| & 0 \end{bmatrix}
$$

$$
\Longrightarrow \begin{bmatrix} 1 & 1+i & \bigm| & 0 \\ 0 & 0 & \bigm| & 0 \end{bmatrix}.
$$

Just as before, we set the variables not corresponding to leading ones to an arbitrary parameter (i.e. $v_2 = t$) so that we get $v_1 = -(1+i)t$. In vector form, and taking $t = 1$, we have

$$
\vec{v}_1 = \begin{bmatrix} -(1+i) \\ 1 \end{bmatrix} = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot i \right).
$$

It should not unexpected that a complex eigenvalue with yield complex eigenvectors. For now, we will we write the solution as a real and an imaginary part without any further consideration. We will see when we revisit differential equations that the real and imaginary parts of a complex eigenvector tell us very powerful things about systems which have strictly *real* variables.

There are a number of notes worth making about this process:

- This is much harder than finding eigenvectors when the eigenvalues were real and distinct! Even in the 2×2 case, this was quite a bit of work. But there is good news—the steps worked exactly as before. That is to say, we can find complex eigenvalues and eigenvectors in exactly the same way as we could for real eigenvalues and eigenvectors, at least in principle. It is just a matter of execution, which takes practice.

- We have not encountered row reduction using complex numbers before. The good news is that the row operations all still hold (i.e. we can multiply by complex numbers, etc.). The bad news is that it is often harder to see what we are supposed to do to eliminate variables (especially if we do not know how to do *division* with complex numbers). A few tricks may help:

1. We know that a complex eigenvalue $\lambda \in \mathbb{C}$ will produce a matrix $(A - \lambda I)$ with rank $< n$ so that there will be (at least one) row of zeroes. So, for a 2×2 matrix, there *must* be a complex number we can multiply by which will show that the two rows are in fact multiples of one another. Just knowing this tells us a great deal about how we must row reduce.

2. A complex number $a + bi$ can always be made into a real number by multiplying by the complex conjugate $a - bi$ since $(a+bi)(a-bi) = a^2 + b^2 \in \mathbb{R}$. If we know two rows must cancel (as they must in the 2×2 case), but the rows have real and complex numbers in places which do not match, a good bet is to multiply by the conjugate of the complex term.

- We might wonder what happened with the conjugate eigenvalue $\lambda_2 = 2 - i$. The answer is that we ignored it, but not without good cause. It turns out that this eigenvalues has the conjugate eigenvector of the one we computed, and that this a general property. That is to say, without even checking, we have know that

$$\vec{v}_2 = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot i \right).$$

Example: Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: We need to compute

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 \\ 1 & 2 - \lambda \end{vmatrix}$$
$$= (1 - \lambda)(2 - \lambda) + 1$$
$$= \lambda^2 - 3\lambda + 3 = 0$$

$$\implies \quad \lambda = \frac{3 \pm \sqrt{9 - 12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i.$$

For complex conjugate eigenvalues, we may pick either eigenvalue to determine the eigenvector, knowing that the other eigenvector will be the

conjugate of the one found. For $\lambda = 3/2 + (\sqrt{3}/2)i$ we have

$$\left[\begin{array}{cc|c} 1 - \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right) & -1 & 0 \\ 1 & 2 - \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right) & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & -1 & 0 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Setting $v_2 = t$, solving in vector form, and then taking $t = 2$, it follows that we have the complex eigenvector

$$\vec{v} = \left[\begin{array}{c} -1 + \sqrt{3}i \\ 2 \end{array}\right] = \left(\left[\begin{array}{c} -1 \\ 2 \end{array}\right] + \left[\begin{array}{c} \sqrt{3} \\ 0 \end{array}\right] \cdot i\right).$$

3   Repeated Eigenvalues

To see what else can happened, let's consider finding the eigenvalues and eigenvectors of

$$A = \left[\begin{array}{cc} -4 & -1 \\ 4 & 0 \end{array}\right].$$

We have

$$\det(A - \lambda I) = \left|\begin{array}{cc} -4 - \lambda & -1 \\ 4 & -\lambda \end{array}\right|$$
$$= (-4 - \lambda)(-\lambda) + 4$$
$$= \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

We do not even need to use the quadratic formula to determine that $\lambda = -2$. We notice that there is something distinctively different about this example since we have only found one eigenvalue instead of the typical two.

10

Nevertheless, we can proceed as before and compute

$$\begin{bmatrix} -4-(-2) & -1 & \Big| & 0 \\ 4 & -(-2) & \Big| & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -2 & -1 & \Big| & 0 \\ 4 & 2 & \Big| & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & \frac{1}{2} & \Big| & 0 \\ 0 & 0 & \Big| & 0 \end{bmatrix}.$$

Setting $v_2 = t = 2$ (to clear the denominator) gives us the eigenvector $\vec{v} = (-1, 2)$.

Something is very different about this example. We have obtained an eigenvector, but only one, and since there are no further eigenvalues to use, this appears to be the end of the discussion. For the applications we will need to use later in this course, however, we will *need* to have a full set of linearly independent eigenvectors—that is to say, we will need as many vectors as there are dimensions to the system. Otherwise we will not be able to find solutions to linear systems of differential equations. But if we are not looking for eigenvectors as we have defined them, what *are* we looking for?

The answer is contained in the following definition.

Definition 3.1. *A vector $\vec{v}_m$ is called a **generalized eigenvector of rank** m if it satisfies*

$$(A - \lambda I)^m \vec{v}_m = \vec{0} \ \ but \ \ (A - \lambda I)^{m-1} \vec{v}_m \neq \vec{0}.$$

It is not immediately obvious how these new vectors help us (and the rigorous justification is beyond the scope of the course), but the following facts are known about generalized eigenvectors:

1. Every defective eigenvalue λ (i.e. an eigenvalue which does not generate as many eigenvectors as its multiplicity in the characteristic equation) has exactly as many generalized eigenvectors as the multiplicity of the λ in the characteristic polynomial (recognizing, of course, that generalized eigenvectors of rank 1 are the regular eigenvectors).

2. These generalized eigenvectors are linearly independent.

3. The generalized eigenvectors form chains of the form

$$(A - \lambda I)\vec{v}_k = \vec{v}_{k-1}, \quad \text{for } k = 1, \ldots$$

11

In fact, this is easy to see! If we have, for instance, $(A - \lambda I)\vec{v}_2 = \vec{v}_1$ where $\vec{v}_1$ is a regular eigenvector (i.e. a vector which satisfies $(A - \lambda I)\vec{v}_1 = \vec{0}$), then we have

$$(A - \lambda I)(A - \lambda I)\vec{v}_2 = (A - \lambda I)\vec{v}_1$$

$$\implies \quad (A - \lambda I)^2 \vec{v}_2 = \vec{0} \ \text{ but } \ (A - \lambda I)\vec{v}_2 \neq \vec{0}.$$

This can be generalized to show that

$$(A - \lambda I)\vec{v}_k = \vec{v}_{k-1}, \quad \text{for } k = 1, \ldots$$

implies
$$(A - \lambda I)^k \vec{v}_k = \vec{0} \ \text{ but } \ (A - \lambda I)^{k-1}\vec{v}_k \neq \vec{0}.$$

Returning to our example, we want to find a generalized eigenvector $\vec{v}_2$ by using the equation
$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

where $\vec{v}_1 = (-1, 2)$. We have

$$\left[\begin{array}{cc|c} -2 & -1 & -1 \\ 4 & 2 & 2 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \right].$$

In vector form, the solution to this system is

$$\left[\begin{array}{c} v_1 \\ v_2 \end{array} \right] = \left[\begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right] + t \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array} \right].$$

As with regular eigenvectors, we may pick any value of $t \in \mathbb{R}$ that we wish, but we may want to take a value which eliminates the fractions. In this case, taking $t = 1$ gives $\vec{v}_2 = (0, 1)$.

For the time being, we will take generalized eigenvectors to be a mysterious quantity, but we may at least be satisfied that this vector, at the very least, is linearly independent of $\vec{v}_1$. Consequently, the set $\{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$. We will see later in the course that determining generalized eigenvectors is necessary for determining the behavior and/or solutions of linear systems of first-order differential equations.

4 Example

Determine the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 3 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: We need to set $\det(A - \lambda I) = 0$. We have

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -\lambda & -2 & 3 \\ 1 & 3 - \lambda & -3 \\ 0 & 0 & 1 - \lambda \end{vmatrix} \\
&= (1 - \lambda)\left[-\lambda(3 - \lambda) + 2 \right] \\
&= (1 - \lambda)\left[\lambda^2 - 3\lambda + 2 \right] \\
&= -(\lambda - 1)^2(\lambda - 2) = 0.
\end{aligned}$$

It follows that $\lambda_1 = 1$ (multiplicity 2) and $\lambda_2 = 2$ are the eigenvalues.

To compute the eigenvector(s) associated with $\lambda_1 = \lambda_2 = 1$, we find the null space of $(A - \lambda_1 I) = (A - (1)I)$. We have

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

It follows that we have $v_2 = s$, $v_3 = t$ and therefore $v_1 = -2s + 3t$. In vector form we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that the eigenvectors associated with λ_1 are $\vec{v}_1 = (-2, 1, 0)$ and $\vec{v}_2 = (3, 0, 1)$.

To compute the eigenvector associated with $\lambda_3 = 2$, we find the null space of $(A - \lambda_3 I) = (A - 2I)$. We have

$$\left[\begin{array}{ccc|c} -2 & -2 & 3 & 0 \\ 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

13

We have $v_2 = t$ so $v_1 = -t$ and also $v_3 = 0$. In vector form, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

It follows that the eigenvector associated with $\lambda_3 = 2$ is $\vec{v}_3 = (-1, 1, 0)$.

We notice that, even though we have a repeated eigenvalue, we did not need to find any generalized eigenvectors! In fact there are *two* regular eigenvectors associated with the single eigenvalue $\lambda = 1$. We need to remember this. *We only need to find generalized eigenvectors if the number of regular eigenvectors is less than the multiplicity of the eigenvalue λ in the characteristic equation.*