MATH 320, WEEK 12:
Higher-Order Differential Equations

1 Higher-Order Differential Equations

We now return to our investigation of differential equations.

In the first differential equation section of this course, we only dealt with *first-order differential equations*. That is to say, we only dealt with equations of the form

$$\frac{dy}{dx} = f(x, y).$$

We saw a few examples where equations of this form arise: population growth models, stirred-tank models, velocity/acceleration problems, etc. But it should come as no surprise that the mathematical models of many physical phenomena *cannot* be represented in this way.

In order to expand the scope of differential equations we can handle, we will need to both understand the physical motivation behind higher-order derivatives and develop the mathematical theory/intuitive required to handle these cases. The key realization, which will not be fully justified until near the end of the course, is that increasing the order of the differential equations, and increasing the number of variables we are considering are in principle the same problem. That is to say, for all intents and purposes, *higher-order* differential equations are the same as *systems* of differential equations. In both settings, we will need to use techniques from linear algebra in order to make progress.

2 Motivation: Damped Spring / Pendulum

We do not need search very hard to find an example of how a *second-order* differential equation may arise in practice. Consider the forces acting on a pendulum (or on an elongated spring). Suppose the rest position is $x = 0$, anything to the right of that is $x > 0$, and anything to the left is $x < 0$. If we move the pendulum to the right ($x > 0$), gravity acts against the pendulum to force it left ($F < 0$); conversely, if we move the pendulum to the left ($x > 0$), gravity acts against the pendulum to force it right ($F > 0$). (See Figure 1.)

If we consider a frictional force in addition to this "restoring force", we have a similar interpretation except in terms of the *velocity*. If we imagine $v = 0$ as no velocity, $v > 0$ as movement to the right, and $v < 0$ as movement to the left, we have that friction always acts *against* the pendulum (i.e. $F < 0$ if $v > 0$ and $F > 0$ if $v < 0$).

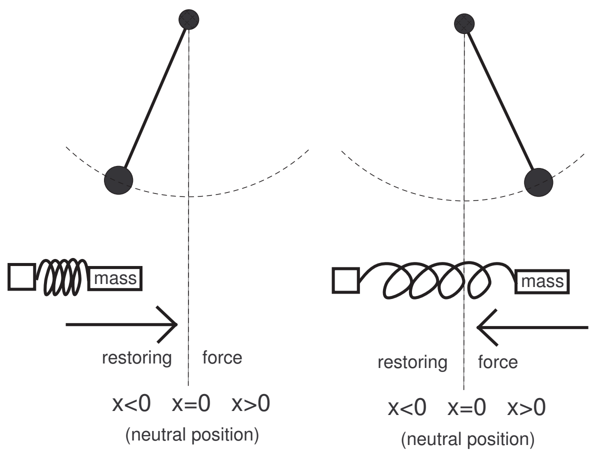

Figure 1: Restoring forces acting on a simple pendulum or a mass-spring. The force acts to restore the mass to its resting or neutral position.

Now let's attempt to capture these forces more precisely. We will assume the following:

1. **Restoring force proportional to position** - That is to say, we will assume that $F_{restoring} = -kx$ for some $k > 0$. This satisfies our previous intuition ($F < 0$ for $x > 0$ and $F > 0$ for $x < 0$) although it is an approximation which does not hold for *high-amplitude* oscillating pendulums (i.e. pendulums that swing very far from the rest position).

2. **Frictional force proportional to velocity** - That is to say, we will assume that $F_{friction} = -cv = -c\frac{dx}{dt}$ for some $c > 0$. This again satisfies our previous intuition. It also makes sense that the more we increase our velocity, the more "drag" we will experience.

The question then becomes how to incorporate this into a differential equation model. The answer comes from Newton's second law $F = ma$ (i.e. force equals mass times acceleration). We have

$$ma = m\frac{d^2x}{dt^2}$$

$$F = F_{restoring} + F_{friction} = -kx - c\frac{dx}{dt}.$$

Putting this all together gives us the combined differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

(Notice that we could derive the same differential equations, with a slightly different interpretation for the constants involved, by considering a mass-spring example obeying Hooke's law.)

This differential equation may not look like much, but it will be our canonical examples (plus or minus a few modifications) for the remainder of the course. There are a few important things to notice about it:

- It is a *second-order* differential equation. It should not take much convincing that the techniques we learned in the early portion of this course (e.g. separating variables, finding integrating factors) are not going to work for finding a solution of such equations (or higher-order equations).

- It is *linear*, *autonomous* and *homogeneous*. In some senses, this is the best possible case, and we will always been able to find solutions. A little later on, we will deal with differential equations like this which are non-homogeneous, i.e. equations like

  $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t).$$

  Recognize, however, that we may consider non-autonomous equations as well, e.g. differential equations

  $$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0.$$

  Even though we will not cover the solution method for such equations, we can still verify that something is a solution. For instance, we can easily compute that $y(x) = x\ln(x)$ is a solution since

  $$y = x, \quad \frac{dy}{dx} = \ln(x) + 1, \quad \frac{d^2y}{dx^2} = \frac{1}{x}, \quad \frac{d^3y}{dx^3} = -\frac{1}{x^2}$$

so that

$$x^3\left(-\frac{1}{x^2}\right) - x^2\left(\frac{1}{x}\right) + 2x(\ln(x)+1) - 2(x\ln(x)) = 0.$$

- There is a further subtlety regarding initial conditions. Consider looking at a snapshot of a pendulum extended to the right and asking the question of what happened to the pendulum in the next moments after the snapshot was taken. We should quickly realize that there are three cases:

  1. If the snapshot was taken while the pendulum was *at rest*, the pendulum will slowly pick up speed from rest and move toward its resting position.
  2. If the snapshot was taken while the pendulum was *swinging to the right*, the pendulum will continue to the right, lose speed, and eventually reverse (or swing over the top).
  3. If the snapshot was taken while the pendulum was *returning from the right*, the pendulum is already moving and will quickly return to the rest position (and probably far exceed it).

  In any case, we see that it is very important to consider not only the *position* of the pendulum at the time the snapshot was taken, but also the *velocity*. In general, for n^{th} order differential equations we need n distinct initial conditions.

Over the next few weeks, we will see how to handle such equations, and how to interpret the results in terms of the relevant physical models as well.

3 Higher-Order Linear Differential Equations

Consider the general homogeneous second-order differential equation with constant coefficients given by

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0. \tag{1}$$

How might we go about finding a solution for such an equation? We cannot separate the variables, or find an integrating factor, or find an obvious substitution which will reduce the differential equation to first-order. So what is there left to do?

The answer is that we are going to *guess*. This may seem like an unsatisfactory answer, but at the very least we will take an educated guess. We know one function which behaves particularly well under the operation of differentiation: the exponential function. So let's guess that the solution of (1) has the form

$$y(x) = e^{rx}$$

for some r. If this does not work out, we have not lost a great deal of time. It is easy to take derivatives of the exponential function!

Example: Find a solution to

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0. \tag{2}$$

Solution: We will guess that the solution has the form $y(x) = e^{rx}$. This gives

$$y = e^{rx}, \quad \frac{dy}{dx} = re^{rx}, \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

so that

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = r^2e^{rx} - 5re^{rx} + 4e^{rx} = e^{rx}(r^2 - 5r + 4) = e^{rx}(r-1)(r-4).$$

The only way for this to equal zero is to have $r = 1$ or $r = 4$. It follows that

$$y_1(x) = e^x, \quad \text{and} \quad y_2(x) = e^{4x}$$

are solutions of the differential equation.

This example raises a very interesting follow-up question: What does it mean for a differential equation to have *two* solutions? Is there only one answer? Or are there two?

The answer is that there is one answer, in a sense, which comes from the two answers we have found. In fact, we can easily check that any *linear combination* of $y_1(x)$ and $y_2(x)$ is a solution of (2)! That is to say, the solution is

$$y(x) = C_1y_1(x) + C_2y_2(x) = C_1e^x + C_2e^{4x}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. This is called the **general solution** of the differential equation.

Well, wait a second... we just finished a section of the course where we were dealing with linear combinations of *vectors*. Now we are considering linear combinations of *functions*? Can we do this? Do the concepts we developed when we were considering vectors spaces (e.g. span, basis, linear independence, etc.) have meaning when we consider solutions of differential equations?

The answer is a definitive *yes*... although we will not go far down the path of extending the vector space notions to functional spaces (in this course). We will, however, need a few crucial concepts, starting with what it means for two functions to be *linearly independent*.

Definition 3.1. *A set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ will be said to be **linearly independent** on the open interval $(a, b) \subset \mathbb{R}$ if*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{3}$$

*for $x \in (a, b)$ implies $c_1 = c_2 = \cdots = c_n = 0$. Conversely, the set is said to be **linearly dependent** on (a, b) if (3) is satisfied for all $x \in (a, b)$ for non-trivial c_i, $i = 1, \ldots, n$.*

Note: For two functions $f_1(x)$ and $f_2(x)$ we have that they are linearly independent if they are not constant multiples of one another. That gives a very easy check for linear dependence for sets of two functions! All we need is $f_2(x) = cf_1(x)$.

It is easy to see that most elementary functions and combinations of elementary functions are linearly independent everywhere they are defined. That is to say, the functions e^x, e^{3x}, $\sin(x)$, $\cos(x)$, x, x^2, $\ln(x)$, etc., are all linearly independent of one another. We will have to be occasionally careful not to get too confident in this property, however. The functions $f_1(x) = \sin(2x)$ and $f_2(x) = \sin(x)\cos(x)$, for instance, *are* linearly dependent (since the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ implies $f_1(x) = 2f_2(x)$ which implies $f_1(x) - 2f_2(x) = 0$).

Just as in the case of vectors, there is an easy check for whether functions are linearly independent. Unsurprisingly, it involves a (very carefully constructed) determinant.

Definition 3.2. *The **Wronskian** of a set of function $f_1(x), f_2(x), \ldots, f_n(x)$ is defined as the $n \times n$ determinant*

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \cdots & f_n^{(n)}(x) \end{vmatrix}.$$

Theorem 3.1. *A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly independent on the interval $(a, b) \subseteq \mathbb{R}$ if $W(f_1, \ldots, f_n) \neq 0$ for some $x \in (a, b)$. Furthermore, if a set of functions is linearly dependent on the interval (a, b) then $W(f_1, \ldots, f_n) = 0$ for all $x \in (a, b)$.*

Note: It should be noted that $W(f_1, \ldots, f_n) = 0$ for all $x \in (a, b)$ *is not* sufficient to conclude that the set $f_1(x), f_2(x), \cdots, f_n(x)$ is linearly dependent, although it is strongly suggestive of the fact!

Example: Show that the functions $f_1(x) = e^x$, $f_2(x) = \sin(x)$ and $f_3 = \cos(x)$ are linearly independent.

Solution: In order to apply Theorem 3.1, we need to compute the Wronskian. We have

$$\begin{aligned}
&\begin{vmatrix} e^x & \sin(x) & \cos(x) \\ e^x & \cos(x) & -\sin(x) \\ e^x & -\sin(x) & -\cos(x) \end{vmatrix} \\
&= e^x \begin{vmatrix} \cos(x) & -\sin(x) \\ -\sin(x) & -\cos(x) \end{vmatrix} - e^x \begin{vmatrix} \sin(x) & \cos(x) \\ -\sin(x) & -\cos(x) \end{vmatrix} \\
&\quad + e^x \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} \\
&= e^x(-1) - e^x(0) + e^x(-1) = -2e^x.
\end{aligned}$$

Since this is everywhere not equal to zero, it follows from Theorem 3.1 that the functions are linearly independent on $\mathbb{R}$ (i.e. the entire real number line).

Now we want to investigate how the concepts of linearly combinations, linear independence, and the Wronskian of functions helps us solve higher-order differential equations. We will consider the following general form of an n^{th}-order linear and homogeneous differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = f(x) \qquad (4)$$

together with the initial conditions

$$y(x_0) = b_0, \quad y'(x_0) = b_1, \quad \ldots \quad y^{(n-1)}(x_0) = b_{n-1}. \qquad (5)$$

Theorem 3.2. *Suppose $f(x) = 0$ and $y_1(x), y_2(x), \ldots, y_m(x)$ are solutions of (4). Then*

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_my_m(x)$$

is a solution of (4).

Proof. We have that

$$y^{(i)}(x) = C_1 y_1^{(i)}(x) + C_2 y_2^{(i)}(x) + \cdots + C_m y_m^{(i)}(x)$$

for all $i = 0, \ldots, n$. We can plug these terms into (4) and collect all the constant C_j terms to get

$$\sum_{j=1}^{m} C_j \left(y_j^{(n)}(x) + p_1(x) y_j^{(n-1)}(x) + \cdots + p_{n-1}(x) y_j'(x) + p_n(x) y_j(x) \right) = 0$$

where we know the bracketting term is equal to zero because all $y_j(x)$ are solutions of (4). It follows that $y(x)$ is a solution and we are done. □

This property is the called the **principle of superposition**. It is worth noting that this property holds very specifically for *linear* and *homogeneous* different equations (i.e. we need $f(x) = 0$), and not for others. For example, the nonlinear differential equation

$$y'(x) - y^{1/2} = 0$$

has the general solution

$$y(x) = \frac{(x - C)^2}{4}.$$

It can be seen, however, that we may not take even a trivial linear combination (i.e. just scaling!) for this function while maintaining the property of being a solution. For instance, the function

$$y_1(x) = 4y(x) = (x - C)^2$$

fails to be a solution because $y'(x) = 2(x - C)$ and $y^{1/2} = x - C$ (for $x \geq C$). It is also necessary for the equation to be *homogeneous*. For instance, the differential equation

$$y'(x) - y = e^x$$

has the general solution

$$y(x) = (x + C)e^x.$$

For instance, it can be easily checked that $y(x) = xe^x$ satisfies the differential equation; however, the function $y_1(x) = 2y(x) = 2xe^x$ does not.

The following theorem tells us what every solution of a linear homogeneous differential equation must look like wherever it is defined.

Theorem 3.3. *Suppose* $f(x) = 0$ *and* $p_1, p_2, \ldots, p_n$ *are continuous on* (a, b). *Then there are* n *linearly independent solutions* $y_1(x), y_2(x), \ldots, y_n(x)$ *of (4). Furthermore, every solution* $y(x)$ *of (4) can be expressed as a linear combination of these functions, i.e. we have*

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

for some real constants $C_1, C_2, \ldots, C_n \in \mathbb{R}$.

This is a terrific result! It tells us that, not only can be combine solutions as linear combinations to form new solutions, but that *every* solution can be constructed in this fashion from some base set of solution. (We could call this set a basis, if we wished.) Notice in particular that the conditions of the theorem are satisfies for the whole number line if we have constant coefficients. It follows that such differential equations have exactly n linearly independent solutions and that every other solution is a linear combination of them.

This tells us properties about the **general solution** of (4) (i.e. the differential equation alone). What if we add some initial conditions? We have the following result.

Theorem 3.4. *Suppose* $p_1, p_2, \ldots, p_n$ *and* f *are continuous on* (a, b). *Then the* n^{th} *order homogeneous differential equations (4) together with initial conditions (5) has a unique solution on the entire interval* (a, b).

This result tells us that, so long as the coefficients are *smooth* functions of the the independent variable (x, in this case), we are guaranteed to have a solution and that they do not bunch up in a certain sense. In particular, if the coefficients are constants, we have exactly one solution which is defined everywhere. A particular realization of a general solution which satisfies some initial conditions is called a **particular solution**.

Note: We do not have the interpretation that each solution through a particular *point* (x, y) is unique (as we did with first-order equations). Rather, we cannot have multiple solutions through a single point (x, y) which is identical in everything up to their $(n-1)^{st}$ derivative. For example, two solutions to a second-order differential equation (4) may go through the same (x, y) point but *must* have a different value of $y'(x)$.

Note: Unless the previous results, we are allowed to consider $f(x) \neq 0$ for this result. We are guaranteed under very general properties that a solution of the form (4) with initial conditions (5) has a solution. It turns out that if $f(x) = 0$ then we also have that the solutions are particularly nice (in that the principle of superposition applies).

This tells us what the solution set looks like—a linear combination of linearly independent functions—but it does not tell us how to *find* these functions. In fact, this can be *very hard* for general linear differential equations with variable coefficients! There are many examples, even second-order examples, where the solution may only be expressed as a power series approximation. That is a topic for another course (Math 319, for instance). For now, we will return to consideration of second-order linear homogeneous differential equations with constant coefficients.

4 Second-Order Linear Differential Equations

Reconsider the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{6}$$

In order to solve this equation, we guess the general form $y(x) = e^{rx}$. We can see very quickly that this yields

$$ar^2e^{rx} + bre^{rx} + e^{rx} = e^{rx}(ar^2 + br + c) = 0.$$

Since $e^{rx} > 0$ for all $x \in \mathbb{R}$, it follows that we must have $ar^2 + br + c = 0$ in order to have a solution. It follows by the quadratic formula that we have

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

This computation should be familiar from computing the eigenvalues of a matrix. It turns out that this is *not* a coincidence—however, we are not quite to the point where we can make the relationship between the two concepts yet. For now, we simply observe that we have three possible cases:

1. If $b^2 - 4ac > 0$ we will have two distinct real values r_1 and r_2.

2. If $b^2 - 4ac = 0$ we will have one repeated real value r.

3. If $b^2 - 4ac < 0$ we will have a complex conjugate pair $r = Re(r) \pm Im(r) \cdot i$.

We have already seen what happens for the first case. The second two cases (as with eigenvalues...) are the trickier cases. They are captured by the following result.

Theorem 4.1. *Consider the second-order linear homogeneous differential equation (6) with constant coefficients. Let r_1 and r_2 be defined by (7). Then the general solution of (6) is:*

1. *If $b^2 - 4ac > 0$ the general solution is $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.*
2. *If $b^2 - 4ac = 0$ the general solution is $y(x) = C_1 e^{rx} + C_2 x e^{rx}$.*
3. *If $b^2 - 4ac < 0$ the general solution is*

$$y(x) = e^{Re(r)x}(C_1 \cos(Im(r)x) + C_2 \sin(Im(r)x)).$$

Proof. We know that we only need to find two linearly independent solutions. We have the following cases:

1. Case 1: If the guess $y(x) = e^{rx}$ produces two distinct real values r_1 and r_2, we have that $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ essentially for free. The only thing remaining is to show that they are linearly independent solutions (check!) to get

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

2. Case 2: If the guess $y(x) = e^{rx}$ only produces the single solution $y_1(x) = e^{rx}$ then we must find another linearly independent solution. Recall that, in order for two solutions to be linearly dependent, we required that $y_2(x) = Cy_1(x)$ (they had to be scalar multiples). For two solutions to be linearly independent, therefore, we must have some variance between $y_2(x)$ and $y_1(x)$. We can represent this difference by introducing another function (not a constant!) $u(x)$ and writing

$$y_2(x) = u(x)y_1(x). \tag{8}$$

This quantifies the fact that $y_2(x)$ and $y_1(x)$ must have more variance between them than just constant multiplication. We want to determine a function $u(x)$ for which the function $y_2(x)$ is a linearly independent solution of (6) given that $y_1(x) = e^{rx}$ is a solution.

We know that $y_1(x) = e^{rx}$ is a solution. This means that $ay_1'' + by_1' + cy_1 = ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$ and, since $b^2 - 4ac = 0$, we have that $r = -b/(2a)$, or $2ar + b = 0$. It follows from (8) that

$$\begin{aligned} y_2(x) &= u(x)y_1(x) = u(x)e^{rx} \\ y_2'(x) &= u'(x)e^{rx} + ru(x)e^{rx} \\ y_2''(x) &= u''(x)e^{rx} + 2ru'(x)e^{rx} + r^2u(x)e^{rx}. \end{aligned}$$

It follows that

$$\begin{aligned} &ay_2''(x) + by_2'(x) + cy_2(x) \\ &= a(u''(x)e^{rx} + 2ru'(x)e^{rx} + r^2u(x)e^{rx}) \\ &\quad + b(u'(x)e^{rx} + ru(x)e^{rx}) + cu(x)e^{rx}. \end{aligned}$$

If we factor the terms containing $u(x)$ we arrive at $u(x)(ar^2e^{rx}+bre^{rx}+ce^{rx}) = 0$ (because e^{rx} is a solution). Furthermore, if we factor the terms containing $u'(x)$, we arrive at $u'(x)(2ar + b)e^{rx} = 0$ (because $r = -b/(2a)$). In order to construct a $y_2(x)$ which is a solution of (6), it is enough to have $ae^{rx}u''(x) = 0$. The only way this can happen is if

$$u''(x) = 0 \quad \Longrightarrow \quad u(x) = Ax + B.$$

It follows that the solution $y_2(x)$ is given by

$$y_2(x) = u(x)y_1(x) = (Ax + B)e^{rx} = Axe^{rx} + Be^{rx}.$$

We now have

$$\begin{aligned} y(x) &= C_1y_1(x) + C_2y_2(x) = C_1e^{rx} + C_2(Axe^{rx} + Be^{rx}) \\ &= (C_1 + C_2B)e^{rx} + C_2Axe^{rx} = \tilde{C}_1e^{rx} + \tilde{C}_2xe^{rx} \end{aligned}$$

and since e^{rx} and xe^{rx} are linearly dependent over $x \in \mathbb{R}$, we are done.

3. Case 3: If the guess $y(x) = e^{rx}$ yields a complex conjugate pair, we have that that $r_{1,2} = Re(r) \pm Im(r) \cdot i$ so that

$$y_{1,2}(x) = e^{Re(r)x \pm Im(r)x \cdot i} = e^{Re(r)x}e^{\pm Im(r)x \cdot i}.$$

This involves the imaginary number $i = \sqrt{-1}$, while we are clearly only interested in real-valued solutions. It turns out that we can use some arithmetic to get rid of the imaginary parts of the equation and find two linearly independent *real-valued* solutions to (6).

It is a well-known fact of complex analysis that

$$e^{xi} = \cos(x) + i\sin(x).$$

This formula is known as **Euler's formula** and can be verified (with a bit of work) by taking the Taylor series expansions of the left- and right-hand sides of the equation. It follows from this that we have the solutions

$$y_1(x) = e^{Re(r)x}e^{Im(r)x\cdot i} = e^{Re(r)x}\left(\cos(Im(r)x) + i\sin(Im(r)x)\right)$$

and

$$y_2(x) = e^{Re(r)x}e^{-Im(r)x\cdot i} = e^{Re(r)x}\left(\cos(Im(r)x) - i\sin(Im(r)x)\right)$$

We know that *any linear combination* of these functions produces a solution of (6). In particular, if we can find a linear combination of these solutions which are *real-valued*, then we are in a much better position. In fact, we can do just that! We notice that

$$\tilde{y}_1(x) = \frac{1}{2}y_1(x) + \frac{1}{2}y_2(x) = e^{Re(r)x}\cos(Im(r)x)$$

and

$$\tilde{y}_2(x) = -\frac{i}{2}y_1(x) + \frac{i}{2}y_2(x) = e^{Re(r)x}\sin(Im(r)x).$$

We have had to take a *complex* linear combination to obtain $\tilde{y}_2(x)$, which may seem a little sneaky, but we are perfectly justified in doing so. The outcome is two solutions which are linearly independent on $x \in \mathbb{R}$ (check!). It follows that the general solution is

$$y(x) = C_1e^{Re(r)x}\cos(Im(r)x) + C_2e^{Re(r)x}\sin(Im(r)x).$$

□

Example 1: Find the general solution of $4y''(x) + 12y'(x) + 9y(x) = 0$. Then find the particular solution for $y(0) = 2$ and $y'(0) = 0$.

Solution: We guess the solution form $y(x) = e^{rx}$. This gives

$$4y''(x) + 12y'(x) + 9y(x) = e^{rx}(4r^2 + 12r + 9) = e^{rx}(2r + 3)^2 = 0.$$

It follows that we only have a solution if $r = -3/2$. Since this is a repeated root, we are in Case 2 and the general solution is given by

$$y(x) = C_1e^{-(3/2)x} + C_2xe^{-(3/2)x}.$$

To solve for the particular solution, we compute

$$y'(x) = -\frac{3}{2}C_1e^{-(3/2)x} + C_2e^{-(3/2)x} - \frac{3}{2}C_2xe^{-(3/2)x}.$$

The conditions $y(0) = 3$ and $y'(0) = 0$ gives the system

$$\begin{aligned} C_1 \qquad &= 2 \\ -\frac{3}{2}C_1 + C_2 &= 0. \end{aligned}$$

We can quickly solve this to get $C_1 = 2$ and $C_2 = 3$. It follows that the particular solution is

$$y(x) = 2e^{-(3/2)x} + 3xe^{-(3/2)x}.$$

Example 2: Find the general solution of $y''(x) + 2y'(x) + 2y(x) = 0$. Then find the particular solution for $y(0) = 1$ and $y'(0) = -1$.

Solution: We guess the solution $y(x) = e^{rx}$. This gives

$$y''(x) + 2y'(x) + 2y(x) = e^{rx}(r^2 + 2r + 2) = 0.$$

The quadratic formula gives the solution

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

Since this a complex root, we are in case 3 and the general solution is

$$y(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x).$$

To solve for the particular solution, we compute

$$\begin{aligned} y'(x) &= -C_1e^{-x}\cos(x) - C_2e^{-x}\sin(x) - C_1e^{-x}\sin(x) + C_2e^{-x}\cos(x) \\ &= -C_1e^{-x}(\cos(x) + \sin(x)) + C_2e^{-x}(\cos(x) - \sin(x)). \end{aligned}$$

The conditions $y(0) = 1$ and $y(0) = -1$ gives the system

$$\begin{aligned} C_1 \qquad &= 1 \\ -C_1 + C_2 &= -1. \end{aligned}$$

It follows immediately that $C_1 = 1$ and $C_2 = 0$ so that the particular solution is

$$y(x) = e^{-x}\cos(x).$$