MATH 320, WEEK 13: Pendulum Model, Non-Homogeneous Linear DEs

1 Pendulum/Spring Model

Let's reconsider the pendulum/spring model from last week. We used Newton's second law $F = ma$ to derive the equation

$$m\frac{d^2x}{dt^2} = F_{restoring} + F_{friction} = -c\frac{dx}{dt} - kx(t) \tag{1}$$

which gives the second-order homogeneous linear differential equation with constant coefficients

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = 0. \tag{2}$$

We can now solve this equation! We can also interpret the solution of this equation. First of all, we have the guess solution $y(x) = e^{rt}$ yields

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - mk}}{2m}$$

The only thing that is different than the general case is that the constants are assumed, for physical reasons, to be strictly positive (i.e. $m > 0$, $c > 0$, $k > 0$).

We have the following three cases:

1. **Overdamped**: If $c^2 > 4mk$ then (1) has the general solution

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

2. **Critically damped**: If $c^2 = 4mk$ then (1) has the general solution

$$x(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

3. **Underdamped**: If $c^2 < 4mk$ then (1) has the general solution

$$x(t) = e^{Re(r)t}(C_1 \cos(Im(r)t) + C_2 \sin(Im(r)t)).$$

Note: The positivity of the physical constants guarantees that either $r < 0$ or $Re(r) < 0$ so long as $c > 0$. (Notice that this is not necessarily true for the general case.) This guarantees that the exponential is always a *decreasing* exponential. This says that the solution is always decaying toward its resting state, as we would expect from a *damped* pendulum or spring.

Note: It is easy to see where the classifications (overdamped, critically damped, and underdamped) come from. In Case (1) the damping parameter exceeds the other combined parameters ($c^2 > mk$), in Case (2) they are equal ($c^2 = mk$), and in Case (3) the other parameters exceed the damping ($c^2 < mk$).

Note on physical units: In order to associate (2) with actual physical models, we will need to give units to the variables and parameters. To make computations as straight-forward as possible, we will consider the units meters (m), kilograms (kg), and seconds (s) for length, mass, and time, respectively. The question, remains, however, of what the units of the *parameters* c and k are. To answer this (as best we can), we recall that a *Newton* is defined as

$$N = 1\ kg\frac{m}{s^2}.$$

This is the basic unit of *force*. We recall that (2) was derived from a force equation—consequently, the unit of each individual term in (2) (i.e. $mx''(t)$, $cx'(t)$ and $kx(t)$) must be Newtons! Since we know the units of $x(t)$ (m), $x'(t)$ (m/s) and $x''(t)$ (m/s^2) we see that the required units for c and k are

$$\begin{aligned} c &\sim kg/s = \frac{kg(m/s^2)}{m/s} = \frac{N}{m/s} \\ k &\sim kg/s^2 = \frac{kg(m/s^2)}{m} = \frac{N}{m}. \end{aligned}$$

Thus we will give the restoring constant k in terms of *Newtons per meter* and the frictional constant c in terms of *Newtons per unit velocity* or *Newtons per meter per second.*

Note on periodic solutions: In solutions to simple mechanical systems, we often encounter the form $C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which represents some sort of periodic motion. What is not obvious from this form, however, is that this is actually equivalent to a *single* phase-shifted trigonometric function with a different amplitude. For instance, we can easily check that

$3\cos(t) + 4\sin(t)$ is the same as $5\cos(t - 0.927)$ (graph it!). In general, we always have that

$$C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A\cos(\omega_0 t - \alpha)$$

for some A and α. We will see how to compute A and α through examples. (You may have already been introduced to the method in a calculus course.)

Although the exponential dominantes the long-term behavior (check by taking the limit as $t \to \infty$!), there are important qualitative differences between the three cases. The difference comes in *how* the solutions approach the resting state.

1. In Case 1, after a short transient period, solutions approach $x = 0$ *monotonically*. That is to say, solutions settle into a trajectory which consistently gets closer as time passes—each second they are closer to $x = 0$ than the last. (Note that trajectories may initial overshoot the resting position if the initial velocity is sufficiently high.)

2. In Case 2, solutions again settled into a trajectory which consistently gets closer to the resting position as time passes, but it takes longer to settle into that trajectory. In fact, it takes the maximal amount of time—if it takes any longer, it will enter into Case 3.

3. In Case 3, solutions *oscillate* as they approach $x = 0$. On average, the solutions approach the resting position, but they continually overshoot the resting position and then bounce back, and overshoot again. Notice that these oscillations continue forever!

The three cases are illustrated by Figure 1. Notice how the exponential dominates in all three cases. Even in Case 3, where solutions oscillate continuously, we may obtain important information about the way in which solutions approach $x = 0$ by bounding by appropriate exponential functions.

Example 1: Consider a 2 kg weight attached to the end of a spring which requires a force of 8 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 2 meters to the right and released with an intial velocity of 2 meters per second to the right, find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form $x(t) = A\cos(\omega_0 t + \alpha)$.

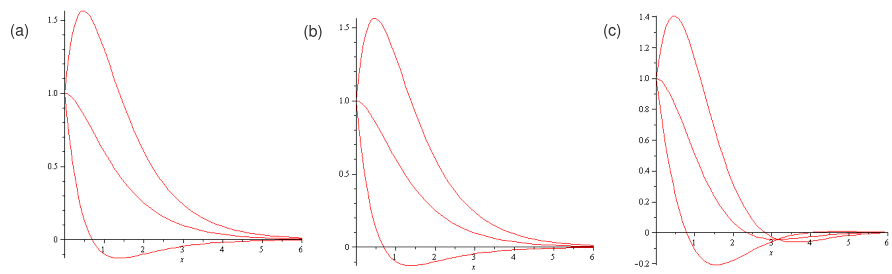

Figure 1: Example trajectories of the displacement of a damped pendulum/spring for the three cases: (a) overdamped; (b) critically damped; and (c) and underdamped. Notice that oscillations occur in the underdamped case.

Solution: The given information implies that $m = 2$, $k = 8$ and $c = 0$. This gives the model

$$2\frac{d^2x}{dt^2} + 8x(t) = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 8) = 2e^{rx}(r^2 + 4) = 0$$

so that $r = \pm 2i$. It follows that the general solution has the form

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

To find the particular solution satisfying the initial conditions, we must compute

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

The initial conditions give

$$\begin{aligned} x(0) = 2 &\quad \Longrightarrow \quad C_1 = 2 \\ x'(0) = 2 &\quad \Longrightarrow \quad 2C_2 = 2 \quad \Longrightarrow \quad C_2 = 1. \end{aligned}$$

It follows that the particular solution is

$$x(t) = 2\cos(2t) + \sin(2t).$$

We want to put the solution in the form $x(t) = A\cos(\omega_0 t - \alpha)$. What we need to do is expand $A\cos(\omega_0 t - \alpha)$ according to

$$A\cos(\omega_0 t - \alpha) = A\cos(\alpha)\cos(\omega_0 t) + A\sin(\alpha)\sin(\omega_0 t).$$

Comparing with the original equations gives $\omega_0 = 2$, and the system

$$\begin{aligned} C_1 &= A\cos(\alpha) \\ C_2 &= A\sin(\alpha). \end{aligned} \tag{3}$$

If we square these equations, add them, and then simplify we get

$$A = \sqrt{C_1^2 + C_2^2}.$$

Furthermore, we can divide the equations to get

$$\alpha = \tan^{-1}\left(\frac{C_2}{C_1}\right).$$

(Note that we may have to adjust α by a factor of π depending on which quadrant it is in. It is a good idea to take the answer here and check with the system (3).)

For our example, we have $C_1 = 2$ and $C_2 = 1$ so that $A = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\alpha = \tan^{-1}(1/2) \approx 0.4636$. We can check that this satisfies $\sqrt{5}\cos(0.4636) = 2$ and $\sqrt{5}\sin(0.4636) = 1$ so that we do not need to adjust by a factor of π. It follows that the solution can be written

$$x(t) = \sqrt{5}\cos(2t - 0.4636).$$

Example 2: Reconsider the set-up provided in Example 1, but assume there is a damping of 4 Newtons for each meter/second of velocity. Find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form $x(t) = A(t)\cos(\omega_0 t + \alpha)$.

Solution: The given information tells us that we have $m = 2$, $k = 8$, and $c = 4$. This gives the model

$$2\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x(t) = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 4r + 8) = 2e^{rx}(r^2 + 2r + 4) = 0$$

which implies

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}i.$$

The general solution is given by

$$x(t) = e^{-t}(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t).$$

In order to determine the particular solution, we must find $x'(t)$. We have

$$\begin{aligned} x'(t) = & -e^{-t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right) \\ & + \sqrt{3}e^{-t}\left(-C_1 \sin(\sqrt{3}t) + C_2 \cos(\sqrt{3}t)\right). \end{aligned}$$

The initial conditions result in the system

$$\begin{aligned} C_1 \qquad\quad &= 2 \\ -C_1 + \sqrt{3}C_2 &= 2. \end{aligned}$$

It follows that $C_1 = 2$ and $C_2 = \frac{4}{\sqrt{3}}$. It follows that the particular solution is

$$x(t) = e^{-t}\left(2\cos(\sqrt{3}t) + \frac{4}{\sqrt{3}}\sin(\sqrt{3}t)\right).$$

A more insightful form of this equation is to write it as

$$x(t) = A(t)\cos\left(\omega_0 t + \alpha\right).$$

As before, we have $A = \sqrt{C_1^2 + C_2^2} = \sqrt{2^2 + (4/\sqrt{3})^2} \approx 3.0551$ and $\alpha = \tan^{-1}(C_2/C_1) = \tan^{-1}((4/\sqrt{3})/2) = \tan^{-1}(2/\sqrt{3}) = 0.8571$. We can easily checked that $3.0551\cos(0.8571) = 2$ and $3.0551\sin(0.8571) = \frac{4}{\sqrt{3}}$ so that we do not need to adjust by a factor of π. It follows that the solution can be written as

$$x(t) = 3.0551e^{-t}\cos\left(2t - 0.8571\right).$$

2 Nonhomogeneous Linear Differential Equations

Suppose now that we take our spring or pendulum but, in addition to the restoring and frictional forces, and apply an external *forcing term* to the system.

For example, consider the simple act of *shaking* a pendulum. Even if we shake the pendulum in a very regular way (say, like a sine function) we will undoubtedly end up with a different solution. But how does the underlying model change? Let's see!

We now want to reconsider Newton's second law but with an addition *independent* forcing term $f(t)$. By independent, we mean that it does not depend on the *state* of the pendulum (position or velocity). The force we add (e.g. shaking) comes from an external source (e.g. our hand, or a motor). In general, we preset the forcing to depend only on the time we look at the system. Together with the restoring and frictional forces, this gives

$$F = F_{restoring} + F_{friction} + F_{forcing} = -kx(t) - c\frac{dx}{dt} + f(t).$$

The resulting differential equation is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = f(t). \tag{4}$$

There are several very important things to note about this equation:

- It is still a second-order linear differential equation with constant coefficients! It should not come as a surprise that the techniques we used to solve such differential equations will be relevant here.
- The external forcing term $f(t)$ ensures that the differential equation is nonhomogeneous. This seems like a small change—and we will see that we will be able to handle this change—but at first glance this is *very bad* news. We are no longer guaranteed that the principle of superposition or decomposition of solutions will hold.

In order to investigate how we might attempt to solve such equations, let's consider a specific example. Suppose we want to solve

$$\frac{d^2x}{dt^2} + 4x(t) = 12t.$$

How can we solve this equation?

The answer may be surprising, but it is not the first time we have heard it. Let's just *guess.* This time, however, let's guess a solution which satisfies the forcing term specific. In other words, we want to guess a *single* solution of the *entire* equation. In fact, this is surprisingly easy to do. We can see immediately that $x(t) = 3t$ is a solution of the equation since it satisfies $4x(t) = 12t$ and $x''(t) = 0$. (Probably the only way we would not have seen this would have been to overthink the problem!)

In fact, finding this solution is sufficient to completely solve the problem! More specifically, once we have found a *particular* solution $x_p(t)$ which satisfies the nonhomogeneous equation (4), we can reduce the problem into a linear *homogeneous* equation.

This is the moral of the following result.

Theorem 2.1. *Consider an n^{th} order linear, nonhomogeneous differential equation of the form*

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + A_1 \frac{dy}{dx} + A_0 y(x) = f(x). \tag{5}$$

Then any solution of (5) can be written

$$y(x) = y_c(x) + y_p(x)$$

where $y_p(x)$ is a particular solution *of (5) and the* complementary function $y_c(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$ *is the solution of the homogeneous system*

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + A_1 \frac{dy}{dx} + A_0 y(x) = 0. \tag{6}$$

Note: This result tells us that, by solving homogeneous linear systems, we have already done most of the work for solving nonhomogeneous linear systems. This is awesome! We only need to worry about this little extra bit and then we are done!

3 Method of Undetermined Coefficients

The question then becomes how we find the particular solution. The answer may not be satisfying, but it is becoming a common one: we are going to guess! We will not be guessing at random, however. We notice that the left-hand side of (5) involves just y and its derivatives. What we need is a form of $y_p(x)$ which can be differentiated to give the form of $f(x)$ on the right-hand side. We notice that

$$\begin{aligned} \frac{d}{dx}[\text{polynomial}] &= \text{polynomial} \\ \frac{d}{dx}[\text{exponential}] &= \text{exponential} \\ \frac{d}{dx}[\text{sine and/or cosine}] &= \text{sin and/or cosine}. \end{aligned}$$

This suggests that we guess trial functions $y_p(x)$ of the following forms:

$$\begin{aligned} y_p(x) &= A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 \\ y_p(x) &= B e^{rt} \\ y_p(x) &= A \cos(ax) + B \sin(ax) \end{aligned} \tag{7}$$

For the respective cases. In order to solve for the constants in the trial function $y_p(x)$, we will need to plug the function into (5).

To summarize, we have the following steps for a linear homogeneous differential equations with constant coefficients (5):

1. Find the general solution $y_c(x)$ of the associated homogeneous equation (6).
2. Select a trial function $y_p(x)$ of some combination of the forms (7) (depending on $f(x)$).
3. Plug the trial function $y_p(x)$ into (5) and solve for the undetermined coefficients.
4. Write the general solution as $y(x) = y_c(x) + y_p(x)$.

This method is fittingly called the **method of undetermined coefficients**.

Example 1: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = e^{-x} - 3x^3.$$

Solution: We need to first solve the homogeneous equation

$$\frac{d^2y}{dx^2} + 4y(x) = 0.$$

The guess $y_c(x) = e^{rx}$ gives $e^{rx}(r^2 + 4) = 0$ so that $r = \pm 2i$. It follows that

$$y_c(x) = C_1 \cos(2x) + C_2 \sin(2x).$$

We now need to use a trial function $y_p(x)$ with a suitable form that it could give $e^{-x} - 3x^3$ after differentiation. We try

$$\begin{aligned} & y_p(x) = Ae^{-x} + Bx^3 + Cx^2 + Dx + E \\ \implies \quad & y_p'(x) = -Ae^{-x} + 3Bx^2 + 2Cx + D \\ \implies \quad & y_p''(x) = Ae^{-x} + 6Bx + 2C. \end{aligned}$$

It follows that the differential equation gives

$$\begin{aligned} & y_p''(x) + 4y_p(x) \\ & = (Ae^{-x} + 6Bx + 2C) + 4(Ae^{-x} + Bx^3 + Cx^2 + Dx + E) \\ & = 5Ae^{-x} + 4Bx^3 + 4Cx^2 + (6B + 4D)x + (2C + 4E) \\ & = e^{-x} - 3x^3. \end{aligned}$$

It follows that we need to satisfying

$$\begin{aligned} 5A &= 1 \\ 4B &= -3 \\ 4C &= 0 \\ 6B + 4D &= 0 \\ 2C + 4E &= 0. \end{aligned}$$

It follows that we have $A = 1/5$, $B = -3/4$, $C = 0$, $D = 9/8$, and $E = 0$. The corresponding particular solution is

$$y_p(x) = \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

The general solution is therefore

$$y(x) = y_c(x) + y_p(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

Note: It is possible that we may need to use more complicated combinations of these functions. For instance, if the forcing term is $e^x \sin(x)$, we will need to use $y_p(x) = Ae^x \cos(x) + Be^x \sin(x)$. A term like $x^2 e^{-x}$ would need $(Ax^2 + Bx + C)e^{-x}$, and so on.

Note: The arguments *inside* the trigonometric and exponential terms are also important. If there are distinct constants, we will need to use distinct trial functions. For instance, the forcing term $f(x) = \sin(2x) + \cos(3x)$ requires the trial function $y_p(x) = A\cos(2x) + B\sin(2x) + C\cos(3x) + D\cos(3x)$.

4 Modifications

Now find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y(x) = \cos(2x).$$

We already have the complementary function $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$. We need to guess the form of the trial function $y_p(x)$. We need terms which can produce $\sin(2x)$ upon differentiation so we choose

$$y_p(x) = A\cos(2x) + B\sin(2x).$$

This gives

$$\begin{aligned} y_p'(x) &= -2A\sin(2x) + 2B\cos(2x) \\ y_p''(x) &= -4A\cos(2x) + 4B\sin(2x). \end{aligned}$$

It follows that we have

$$\begin{aligned} &y_p''(x) + 4y_p(x) \\ &= -4A\cos(2x) + 4B\sin(2x) + 4(A\cos(2x) + B\sin(2x)) = 0. \end{aligned}$$

We need to match constants so that this equals $f(x) = \sin(2x)$ but the term has vanished. We have nothing left to work with! Something has gone terribly wrong, but what?

We might notice that we should have expected this. After, the complementary function is $y_c(x) = C_1\cos(2x) + C_2\sin(2x)$, which meant we know that the combination of functions in the trial function *had to* vanish when it was substituted into the left-hand side of our differential equation. This raises a very important concern which we will have to identify:

- The trial functions (7) will only work if the individual functions *do not* appear in the complementary function $y_c(x)$.

In other words, if a forcing term coincides with a term already contained in the dynamics of the unforced system, we will not be able to construct nontrivial trial function in the same way as we have been. (This will have a very important interpretation when we return to the consideration of our physical pendulum/spring model!)

It turns out that in this case we will have to use *different* trial functions. What we really need to do is generate *other* linearly independent solutions. We have already done this! For instance, we found that if we had a solution $y_1(x) = e^x$ and needed another linearly independent one, that we could use $y_2(x) = xe^x$. If we need another one, we went up to $y_3(x) = x^2e^x$, and so on.

The same trick will work here! We will take our trial functions to be the same as used in (7) but with as many powers of x are required to give a *linearly independent* function. More succinctly, if the terms in the trial functions (7) appear in the complementary function $y_c(x)$, we must instead use the trial functions

$$\begin{aligned} y_p(x) &= A_nx^{n+s} + A_{n-1}x^{n+s-1} + \cdots + A_1x^{s+1} + A_0x^s \\ y_p(x) &= Bx^se^{rt} \\ y_p(x) &= Ax^s\cos(ax) + Bx^s\sin(ax) \end{aligned} \tag{8}$$

where s is the lowest power which produces a term which is linearly independent of those in the complementary solution.

Example: Reconsider the example

$$\frac{d^2y}{dx^2} + 4y(x) = \cos(2x).$$

The complementary function was $y_c(x) = C_1\cos(2x) + C_2\sin(2x)$ so we are not allowed to use $y_p(x) = A\cos(2x) + B\sin(2x)$ as a trial function. Instead, we must use

$$y_p(x) = Ax\cos(2x) + Bx\sin(2x).$$

This gives

$$\begin{aligned} y_p'(x) &= A\cos(2x) + B\sin(2x) - 2Ax\sin(2x) + 2Bx\cos(2x) \\ y_p''(x) &= 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x). \end{aligned}$$

Plugging into the DE gives

$$\begin{aligned} &y_p'' + 4y_p \\ &= 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x) \\ &\quad + 4(Ax\cos(2x) + Bx\sin(2x)) \\ &= 4B\cos(2x) - 4A\sin(2x) \\ &= \cos(2x). \end{aligned}$$

It follows that we need $A = 0$ and $B = 1/4$ so that we have the particular solution

$$y_p(x) = \frac{1}{4}x\sin(2x).$$

The general solution of the differential equation is therefore

$$y(x) = C_1\cos(2x) + C_2\sin(2x) + \frac{1}{4}x\sin(2x).$$

5 Forced Mechanical Systems and Resonance

Let's consider what happens for the undamped (low-amplitude) pendulum and spring model when the forcing is *sinuisoidal*—that is to say, when the forcing can be represented by some combination of sines and cosines. This represents shaking the undamped pendulum or spring with a fixed frequency.

Consider the example of solving the following differential equation (corresponding to the physical model) with initial conditions $x(0) = 0$ and $x'(0) = 0$:

$$\frac{d^2x}{dt^2} + 4x(t) = \cos(\omega t)$$

where $\omega \neq 2$. (This corresponds to the pendulum system for a mass of 1 kg and a restoring constant k of 4 Newtons per meter. The parameter ω controls the frequency of the shaking and the initial conditions correspond to starting the system at rest.)

We can have already seen that the complementary function for this differential equation is

$$x_c(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

Since $\omega \neq 2$, we use the trial function $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$. This gives

$$x_p''(t) = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)$$

so that we have

$$\begin{aligned} x_p''(t) &= -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t) + 4(A\cos(\omega t) + B\sin(\omega t)) \\ &= (4 - \omega^2)(A\cos(\omega t) + B\sin(\omega t)) \\ &= \cos(\omega t). \end{aligned}$$

Since $\omega \neq 2$ implies $\omega^2 \neq 4$, it follows that $A = 1/(4 - \omega^2)$ and $B = 0$ so that we have the general solution

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4 - \omega^2} \cos(\omega t).$$

This has derivative

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - \frac{\omega}{4 - \omega^2} \sin(\omega t)$$

so that the initial conditions $x(0) = 0$ and $x'(0) = 0$ give the system

$$\begin{aligned} C_1 &= -\frac{1}{4 - \omega^2} \\ 2C_2 &= 0 \end{aligned}$$

which implies $C_1 = -1/(4 - \omega^2)$ and $C_2 = 0$. It follows that the solution is

$$\begin{aligned} x(t) &= -\frac{1}{4 - \omega^2} \cos(2t) + \frac{1}{4 - \omega^2} \cos(\omega t) \\ &= \frac{1}{4 - \omega^2} \left(\cos(\omega t) - \cos(2t)\right). \end{aligned}$$

In terms of simplification, this is pretty good, but in fact we can do a little better. The trigonometric identities $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ can be subtract from one another to give $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$. If we take $A = \frac{1}{2}(2+\omega)t$ and $B = \frac{1}{2}(2-\omega)t$ we have

$$A + B = 2t, \quad \text{and} \quad A - B = \omega t.$$

Remarkably (but usefully?), this implies that the solution can be written as the single term

$$x(t) = \frac{2}{4-\omega^2} \sin\left(\frac{1}{2}(2+\omega)t\right) \sin\left(\frac{1}{2}(2-\omega)t\right).$$

Okay, this is getting a little ridiculous. What is the point of all this algebra? Well, this is actually *incredibly* insightful for of the solution. We now have the solution decomposed into two sine functions with different frequencies (corresponding to the difference in the natural and forcing frequencies!). In particular, if ω is near 2, there is a separation of time-scales in the two modes. We have that

1. There is a *slow* oscillatory mode with wavelength $(4\pi)/(2-\omega)$. This mode can be thought of as an envelop which restricts all other modes (since all other modes must multiply through this function, so can only be as big as this slow mode allows it to be). (See Figure 2(a))

2. There is a *fast* oscillatory mode with wavelength $(4\pi)/(2+\omega)$. This mode oscillates faster than the other mode but is restricted through each period by its slower counterpart.

3. Since sine is bound by -1 and 1, the maximal amplitude of the solution is $2/(4-\omega^2)$.

This raises a very interesting question: What happens as the forcing frequency is *changed* related to the fixed natural frequency of the system (i.e. the frequency the undamped pendulum or spring swings when left alone)? In particular, what happens as $\omega \to 2$?

We can consider this as ω approaches 2 from either side, since the separation of time-scales holds. We make the following observations:

1. The ω approachs 2, the wavelength of the slow mode *explodes* while the wavelength of the fast mode stays roughly the same. That is to say, the separation in time-scales intensifies in that the number of times the fast mode completes its cycle before the slow mode completes its cycle becomes unbounded. (See Figure 2(b).)

2. The amplitude $2/(4-\omega^2)$ also *explodes*. In fact, in the limit, we have that the amplitude is infinite. (See Figure 2(c).)

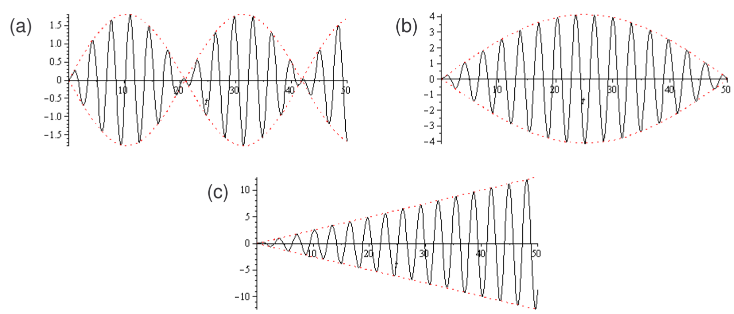

Figure 2: Solution of the mechanical system with sinusoidal forcing with (a) $\omega = 1.7$; (b) $\omega = 1.875$; and (c) $\omega = 1.99$. Notice that the y-axis grows as ω gets closer and closer to 2.

Something seems to be going incredibly wrong in this example. How can we have the amplitude of our solution explode to infinity? Worst still, we know that the solution still oscillates by a fixed period, so as time goes on and one the solution (i.e. the pendulum or spring) will make jumps from the positive extreme to the negative extreme *in the same amount of time*! What is going on?

Let's reconsider our physical example. What is really happening as ω approaches 2? Recall that 2 is the natural frequency term for the *underlying* system. Is the term controlling how the body naturally oscillates if simply let go. Now imagine shaking that in a very particular way—and in particular, in a way that is completely *in phase* with the natural rhythm of the body. Well, then, every time the pendulum naturally wants to kick left, we give it an extra push, and every time it wants to kick right, we give it an extra push in that direction, too. If we do this exactly in sync with the body's natural rhythm, we imagine that the amplitude will certainly grow!

Before we get carried away with this example too far, we should recognize that there are certain physical constraints (e.g. damping, whether in the form of friction or something else). We also neglected other physical concerns. For a pendulum, for example, we will swing over the top far before we will extend off to infinite in any diretion. And, for a spring, we imagine that if we compress or overextend the spring too much it will simply *break* rather than extend to infinite length. Nevertheless, this is an interesting phenomenon to investigate and is a concern in many applications. What we have discovered is **resonance**.

We might wonder what has happened with our solution. After all, we cannot very well have $x(t) = \infty$ as a meaningful solution. Rather, the solution breaks down, but if we consider the original differential equation we immediately see why. If we have $\omega = 2$ we are in the case where we may not use a trial function of the form $x_p(t) = A\cos(2t) + B\sin(2t)$. We have already solved this using the trial function $x_p(t) = At\cos(2t) + Bt\sin(2t)$ and got

$$x(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}t\sin(2t).$$

The initial conditions $x(0) = 0$ and $x'(0) = 0$ give $C_1 = C_2 = 0$ give the simple solution

$$x(t) = \frac{1}{4}t\sin(2t).$$

Just as we expected, we have a solution which oscillates with increasing amplitude (as t grows). In other words, we have filled in the gap in our previous physical reasoning. Even though the solution methods were completely different, the limit of the previous solution approaches this resonate solution as ω approaches 2!