MATH 320, WEEKS 14 & 15: First-Order (Linear) Systems of Differential Equations

1 Motivating Example

A few months ago, we imagined a mathematical model for the chemical reaction $X \to Y$ subject to continuous inflow of X and outflow of both X and Y. If we let the concentrations of X and Y be denoted $x = [X]$ and $y = [Y]$, we imagined the following interactions:

1. Inflow to X at constant rate (say α), and outflow (through conversion to Y and outflow from tank) proportional to current concentration of X (say βx).

2. Inflow to Y (through conversion from X) proportional to current concentration of X (say γx), and outflow (through outflow from tank) proportional to current concentration of Y (say δy)

Making suitable numerical choices for the proportionality constants ($\alpha = \beta = 1$ and $\gamma = \delta = 2$), we arrived at the following differential equation model:

$$\begin{aligned} \frac{dx}{dt} &= 1 - x \\ \frac{dy}{dt} &= 2x - 2y. \end{aligned} \tag{1}$$

Although we did not classify it as such at the time, this is an example of a *first-order system of linear differential equations*, which is the final topic of the course. Before we explicitly state our objectives and methods for this topic area, there are a few things worth noting about this specific model:

- While we still only have one independent variable (t), we now have *two* dependent variables (x and y). Since there are two variables which depend on t, finding a solution of (1) means finding *both* a $x(t)$ and a $y(t)$ which satisfy the differential equations! Similarly, checking or verifying a solution means checking that *both* given functions satisfy the DEs. For this example, we can easily check that $x(t) = 1 - e^{-t}$ and $y(t) = 1 - 2e^{-t} + e^{-2t}$ is a solution because

$$\frac{dx}{dt} = \frac{d}{dt}\left[1 - e^{-t}\right] = e^{-t} = 1 - [1 - e^{-t}] = 1 - x(t)$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left[1 - 2e^{-t} + e^{-2t}\right] \\ &= 2e^{-t} - 2e^{-2t} \\ &= 2[1 - e^{-t}] - 2[1 - 2e^{-t} + e^{-2t}] \\ &= 2x(t) - 2y(t).\end{aligned}$$

- The equations can still be classified in the exact same way as we classified them before. For instance, we will say that this system of DEs is *first-order* because the highest derivative (of either x or y!) is first-order, that it is linear because all the terms we are solving for and their derivatives (i.e. x, y, x', and y') appear isolated from one another, and that it is nonhomogeneous because not every term involves an x, a y, or one of their derivatives (because of the 1 in the first equation).

- We were able to solve this system at the time by noticing that the first equation depended on x only, and not y, and consequently we could solve for $x(t)$ *before* consider $y(t)$. This is *not* a general property of systems of differential equations! In general, the variables are interdependent—that is to say, the equation for x' depends on x *and* y, and the equation for y' depends on x and y as well. In this case, we cannot hope to solve for one variable first and then substitute into the remaining equation. We will have to develop more sophisticated machinery!

2 General First-Order Systems

In general, we can consider systems of differential equations of *arbitrary* order. For instance, if we considered a coupled pendulum system (say, one pendulum hanging below another), we would end up considering a *system* of coupled (i.e. interdependent) *secord-order* differential equations. That is to say, we would arrive at a pair of equations of the form $x'' =$ stuff and $y'' =$ stuff.

All the theory and methodology we develop, however, will be geared toward first-order systems, i.e. systems where the highest derivative in any dependent variable is a first-order derivative. That is to say, we will be considering systems of differential equations of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, x_2, \ldots, x_n).
\end{aligned} \tag{2}$$

We might think that we are losing some important information by restricting ourselves to such an idealized set of equations, but in fact the methods for analysing first-order systems are actually sufficient to (almost always!) analyse higher-order systems. The reason is that higher-order systems differential equations can (almost always!) be rewritten as a system of first-order differential equations by an appropriate change of variables. We have the following general methodology:

- Consider a general n^{th} order differential equation written in the form
$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)). \tag{3}$$

- Make the variable substitutions $x_1(t) = x(t)$, $x_2(t) = x'(t)$, $x_3(t) = x''(t)$, $\ldots$, $x_n(t) = x^{(n-1)}(t)$.

- Notice that we have $x_1'(x) = x_2(t)$, $x_2'(t) = x_3(t)$, $\ldots$. In general, we have $x_i'(t) = x_{i+1}$ for $i = 1, \ldots, n-1$.

- Notice that this only works for the first $n-1$ variables. For the final n^{th} function $x_n(t)$ must return to the original system. We notice that (3) implies $x_n'(t) = f(t, x_1(t), x_2(t), \ldots, x_n(t))$.

- The *system* of *first-order* differential equations corresponding to (3) is
$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_3 \\
&\vdots \\
x_{n-1}' &= x_n \\
x_n' &= f(t, x_1, x_2, \ldots, x_n).
\end{aligned} \tag{4}$$

- The initial conditions $x(t_0) = b_1,\ x'(t_0) = b_2,\ \ldots,\ x^{(n-1)}(t_0) = b_n$ become
$$x_1(t_0) = b_1,\ x_2(t_0) = b_2,\ \ldots\ x_n(t_0) = b_n.$$

Example: Rewrite the initial value problem

$$x''(t) + 4x'(t) + 4x(t) = \sin(2t) \tag{5}$$

$$x(0) = 5,\ x'(0) = -1 \tag{6}$$

as an initial value problem for a system of first-order differential equations.

Solution: We make the substitutions $x_1(t) = x(t)$ and $x_2(t) = x'(t)$. This gives the relationship $x_1'(t) = x_2(t)$. In order to find an equation for $x_2'(t)$ we need to rewrite (5) in the form

$$x''(t) = \sin(2t) - 4x'(t) + 4x(t).$$

We can see that this corresponds to the required form $x''(t) = f(t, x(t), x'(t))$ which becomes $x_2'(t) = f(t, x_1(t), x_2(t))$. The desired system of first-order differential equations is therefore

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 4x_1 - 4x_2 + \sin(2t) \end{aligned}$$

and the initial conditions $x(0) = 5$ and $x'(0) = -1$ become

$$x_1(0) = 5,\ x_2(0) = -1.$$

We have successfully transformed the original *second-order* differential equation into a system of *two* first-order differential equations!

While this process may not seem like much, it is the first step toward defining a general methodology for studying differential equations for very complicated systems! We may shrug off the importance of first-order differential equations, but there are several very important notes to make:

- Notice that the order of the original system (3) corresponds to the number of variables in (4). This is a general propery (although, if there is a system of higher-order equations, the number of overall variables in the complete first-order system formulation will be the sum over all of the individual orders). For example, a 12^{th} order differential equation in x will become a first-order system with exactly 12 variables.

- In some sense, the initial value problem for the system formulation (4) makes more sense than the original set-up (3). In order to completely solve the system (3), we require initial conditions on $x(t)$ and all of its derivatives up to the $(n-1)^{st}$ order. This was a little strange—after all, why does a 7^{th} order equation require exactly 7 initial conditions, and not 5 or 12? In the setting of (4), however, this becomes clear. We require initial conditions on $x_i(0)$, $i = 1, \ldots, n$, because there are exactly n independent variables we need to solve for. If we do not specify where we start for each variable, we will have no hope of figuring out where we go from there!

- Notice that the example system (5) corresponds to a forced spring/pendulum example. We will revisit such equations later. For now, notice that the physical importance of the new variables: $x_1(t)$ corresponds to the *position* of the mass $x(t)$, and $x_2(t)$ corresponds to the its *velocity* $x'(t)$. In order words, we have constructed a first-order system in the position and velocity of the mass!

- There are strong geometric reasons to prefer first-order systems to higher-order differential equations. The first derivative is very easy to interpret graphically—it is the slope of the solution function at the given point. For example, if we have $x'(t_0, x_0) = 1$ at a particular point (t_0, x_0), we know that the slope of the solution through (t_0, x_0) is 1. But what does it mean to have $x''(t_0, x_0) = 1$? It means something to do with the concavity of the solution, for sure, but it would take some significant thought to construct a meaningful picture out of this information (i.e. something like a slope diagram).

  If reformulate the problem as a *system* of first order differential equations, however, we avoid this problem! Instead of having to interpret the *second-order* derivative, we have to consider the slope of *two* variables. This interpretation will allow us to (easily!) draw pictures corresponding to higher-order systems and also to develop numerical methods for estimating solutions when solutions cannot be solved for explicitly (beyond the scope of this course).

3 First-Order Linear Homogeneous Systems of Differential Equations in Two Variables

We start by consider the baseline equations with which we will be working: a system of two linear, first-order homogeneous differential equations with constant coefficients:

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{aligned} \tag{7}$$

Before considering how we might *find* an analytic solution $(x(t), y(t))$ to such a system, let's first ask a more basic question: What *can* a system like this do? Let's consider this question for a geometrical point of view; in other words, let's try to draw a *picture*. We make the following observations:

- The system is first-order so that, at every point (x_0, y_0) in the (x, y)-plane we know whether the solution through the point (x_0, y_0) is pointed right or left ($x'(t) > 0$ or $x'(t) < 0$) or up or down ($y'(t) > 0$ or $y'(t) < 0$).

- We know the equation $x'(t) = 0$ corresponds to $ax + by = 0$ or $y = -(a/b)x$ and $y'(t) = 0$ corresponds to $y = -(c/d)x$. In other words, we know exactly where the solution $(x(t), y(t))$ is completely flat or completely straight up/down.

Example 1: Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -x + 3y \\ \frac{dy}{dt} &= 3x - y. \end{aligned}$$

We can easily determine that

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \quad y = \frac{1}{3}x$$

and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad y = 3x.$$

The question then becomes what happens in the regions between these two lines. It should not take too much convincing that, if we only consider arrows pointing in the dominant directions (NW, NE, SW, SE, say) that we arrive at the picture given by Figure 1(a).

Example 2: Consider the system of differential equations

$$\frac{dx}{dt} = -x + 5y$$
$$\frac{dy}{dt} = -2x + y.$$

We can easily determine that $x' = 0$ implies $y = (1/5)x$, and that $y' = 0$ implies $y = 2x$. When we consider the orthants, we end up with a picture that looks something like Figure 1(b).

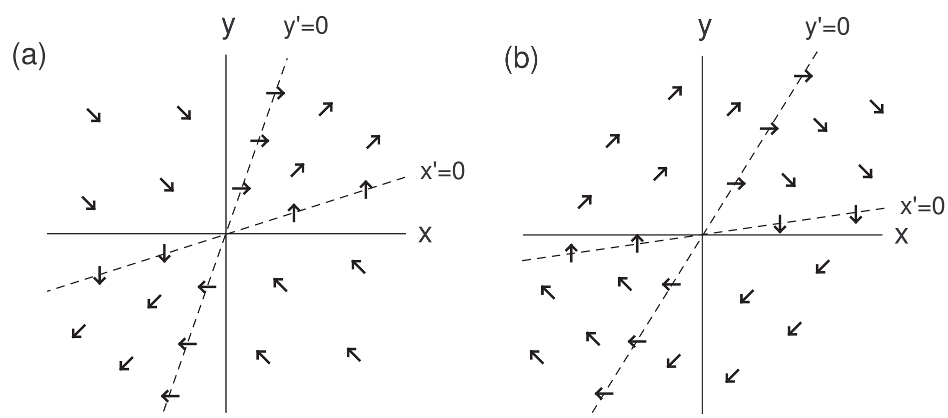

Figure 1: A rough sketch of the two example systems. Even without solving the equations, we can get some sense about how the solutions behave.

Without even attempting to solve the system of differential equations, we can tell very important things about the types of behaviors we might encounter. It looks like the solutions of the first system originate somewhere in the top-left or bottom-right, pool together, then travel toward either the top-right or the bottom-left. Solutions of the second system, by contrast, appear to spiral around $(0,0)$.

4 Solutions to Linear Systems

In order to investigate how we might find a solution to these systems, or a general system such as (7), let's make an observation about the form of the equation. Notice that the right-hand side is something we have already seen, but not in the differential equation portion of the course. In fact, it is exactly the type of thing we saw when consider *linear systems of equations*. In that section of the course, we found that linear systems could be written in the form $A\vec{x} = \vec{b}$, i.e. that the system part could be condensed by making use of matrix multiplication.

We can do the same thing here! The form will be a slightly different than in the previous setting, because we are dealing with unknown functions rathern than unknown variables, but all the previous intuition implies. We can write the system (7) as

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{8}$$

where

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \; A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; \text{ and } \; \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This form suggests immediately that the tools we developed from linear algebra will be relevant for solving *systems* of differential equations. In fact, this intuition is completely justified—we will see that we already have all of the tools needed to completely solve systems of DEs of the form (8)!

Before we get there, however, let's try to build some intuition. The first order equation (8) is a vector/matrix equation, but it looks eerily similar to the first order equation

$$\frac{dx}{dt} = ax$$

which we know has solution $x(t) = Ce^{at}$. The question then becomes, can we extend our standard algebra result by substituting matrix algebra instead? What are the terms going to be? Can we write e^{At} for a matrix A? (We can, but won't!) Is there some other way we can extend the solution $x(t)$ to the vector solution $\vec{x}(t)$?

Consider the following set-up. We guess a solution $\vec{x}(t)$ with the general exponential form e^{at}, but we allow the components of $\vec{x}(t)$ to vary according to a pre-defined vector. In other words, we write $x(t) = \vec{v}e^{\lambda t}$ for some $\lambda \in \mathbb{R}$. This keeps the general intuition that the solution is exponential while allowing us to extend to a vector form.

Now let's check the matrix equation (8)! We have

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}[\vec{v}e^{\lambda t}] = [\lambda\vec{v}]e^{\lambda t}$$

and

$$A\vec{x} = A[\vec{v}e^{\lambda t}] = [A\vec{v}]e^{\lambda t}.$$

It follows that we need

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \Longrightarrow \quad [\lambda\vec{v}]e^{\lambda t} = [A\vec{v}]e^{\lambda t}.$$

After dividing by $e^{\lambda t}$ (which is never zero) and rearranging, we have

$$A\vec{v} = \lambda\vec{v}.$$

If you get the sense that we have seen this equation before, it is because we have—but it previously had nothing to do with differential equations at all. This is the eigenvalue/eigenvector equation for the matrix A, the equation which told us the invariant directions of A treated as a linear transformation. Now it tells us something different: it tells us that eigenvalues λ_i and eigenvectors $\vec{v}_i$, $i = 1, \ldots, n$, give us solutions to (8) of the form $\vec{x}_i(t) = \vec{v}_i e^{\lambda_i t}$.

Reconsider the example $x' = -x + 3y$ and $y' = 3x - y$, we have

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{with} \quad A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

We can quickly compute that the eigenvalues are given by $(-1-\lambda)(-1-\lambda) - 9 = \lambda^2 + 2\lambda - 8 = (\lambda+4)(\lambda-2) = 0$ so that $\lambda_1 = -4$ and $\lambda_2 = 2$. The corresponding eigenvectors are $\vec{v}_1 = (1, -1)$ and $\vec{v}_2 = (1, 1)$. It follows that we have two solutions of the form $\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t}$ and $\vec{x}_2(t) = \vec{v}_2 e^{\lambda_2 t}$. In fact, we can take any linear combination of these solutions. The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Remarkably, knowing how to compute eigenvalues and eigenvectors completely solves the problem! We also get to complete our earlier picture. Since $C_1(1, -1)e^{-4t} \to 0$ as $t \to \infty$, we have that the solution gets closer to $C_2(1, 1)e^{2t}$ as time goes on. In other words, solution approach the line spanned by $(1, 1)$. This is where the arrows were points in the picture in Figure 1(a).

Of course, things are not always quite so easy. Ignoring some subtleties, there are three basic cases for eigenvalues: distinct real eigenvalues, repeated eigenvalues, and complex eigenvalues. The solution forms for the latter two cases, of course, are different for the repeated eigenvalue and complex eigenvalue cases. But we have already how they come about! If we have a repeated eigenvalue, without a spanning set of eigenvectors, then we do not get a full set of linearly independent solutions to the system—but we have already seen how to generate linear independent solutions for repeated roots. Similarly, complex eigenvalues $\lambda = \alpha \pm \beta i$ lead to two linearly independent solutions involving $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$.

In any case, the solution for the 2×2 differential equation cases can be completely determined by the eigenvalues and eigenvectors in the following way:

1. **Two real distinct eigenvalues** (or a **repeated eigenvalue with two distinct eigenvectors**) - If we have two distinct eigenvalues λ_1 and λ_2 corresponding to $\vec{v}_1$ and $\vec{v}_2$, respectively, the solution to (8) is given by
$$\vec{x}(t) = C_1\vec{v}_1e^{\lambda_1 t} + C_2\vec{v}_2e^{\lambda_2 t}.$$
Similarly, if there is a repeated eigenvalue ($\lambda = \lambda_1 = \lambda_2$) but two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$, we have
$$\vec{x}(t) = e^{\lambda t}\left(C_1\vec{v}_1 + C_2\vec{v}_2\right).$$

2. **Repeated eigenvalue, one eigenvector** - If we have a repeated eigenvalue $\lambda = \lambda_1 = \lambda_2$ but only one eigenvector $\vec{v}$, we have the general solution
$$\vec{x}(t) = (C_1\vec{v} + C_2(t\vec{v} + \vec{w}))e^{\lambda t}$$
where $\vec{w} \in \mathbb{R}^2$ is a **generalized eigenvector** satisfying
$$(A - \lambda I)\vec{w} = \vec{v}.$$

3. **Complex eigenvalues** - If we have a complex eigenvalue $\lambda = \alpha + i\beta$ corresponding to a complex eigenvector $\vec{v} = \vec{a} + i\vec{b}$ then the general solution is given by
$$\begin{aligned}\vec{x}(t) = C_1e^{\alpha t}&\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right)\\ &+ C_2e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right).\end{aligned}$$

There are a few notes worth making about these solutions:

1. It is clear that exponentials factor very heavily in the solutions of linear systems of differential equations. We also notice that, in terms of limiting behavior, these exponentials dominate the behavior (i.e. they asymptotically overwhelm the factor t in case (2), and the trigonometric functions in (3)). That is to say, the long-term behavior is determined by the exponentials, so that trajectories tend to decay (i.e. approach $(0,0)$) if $\mathrm{Re}(\lambda) < 0$ and blow up (i.e. go away from $(0,0)$) if $\mathrm{Re}(\lambda) > 0$.

2. The case when $\lambda = 0$ is somewhat special, but it is worth noting that the solution forms for case (1) and (2) still hold, but that the exponential becomes one.

Example 1: Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= -x + 3y, \qquad x(0) = 1 \\ \frac{dy}{dt} &= 3x - y, \qquad y(0) = 1. \end{aligned}$$

We have already determined that the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

It remains to use the initial conditions to solve for C_1 and C_2. We have that $x(0) = 1$ and $y(0) = 1$ so that at $t = 0$ we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can rewrite this as

$$\begin{aligned} -C_1 + C_2 &= 1 \\ C_1 + C_2 &= 1 \end{aligned}$$

or, equivalently,

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can easily row-reduce this to get

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right].$$

It follows that $C_1 = 0$ and $C_2 = 1$ so that the particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Example 2: Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= -x + 5y, \qquad & x(0) = 1 \\ \frac{dy}{dt} &= -2x + y, \qquad & y(0) = 1. \end{aligned}$$

To find the eigenvalues, we realize

$$A = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} -1-\lambda & 5 \\ -2 & 1-\lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(-1-\lambda)(1-\lambda) + 10 = \lambda^2 + 9 = 0.$$

It follows that $\lambda = \pm 3i$. We need to find the eigenvectors corresponding to these values. We have

$$(A - (3i)I) = \begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix}.$$

To find the nullspace, we row reduce to get

$$\left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right] \xrightarrow{(-1+3i)R_1} \left[\begin{array}{cc|c} (-1-3i)(-1+3i) & 5(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 10 & -5+15i & 0 \\ -2 & 1-3i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2}+\frac{3}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $\vec{v} = (1-3i, 2)$. We rewrite this as

$$\vec{v} = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

We set $\alpha = Re(\lambda) = 0$ and $\beta = Im(\lambda) = 3$ and $\vec{a} = Re(\vec{v}) = (1, 2)$ and $\vec{b} = Im(\vec{v}) = (-3, 0)$. It follows that the general solution is

$$\begin{aligned} \vec{x}(t) = C_1 &\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &+ C_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right). \end{aligned}$$

To solve for C_1 and C_2, we utilize the initial conditions $x(0) = 1$ and $y(0) = 1$. At $t = 0$ we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

so that we have

$$\begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It follows immediately that $C_1 = 1/2$ and $C_2 = -1/6$ so we have

$$\begin{aligned} \vec{x}(t) &= \frac{1}{2}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &\quad - \frac{1}{6}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(3t) + \frac{1}{3} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \sin(3t) \end{aligned}$$

Example 3: Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= x - 4y, \qquad & x(0) &= -1 \\ \frac{dy}{dt} &= x - 3y, \qquad & y(0) &= 2. \end{aligned}$$

To find the eigenvalues, we realize

$$A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(1-\lambda)(-3-\lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

so that $\lambda = -1$ is a repeated eigenvector. To check for the eigenvector(s) corresponding to this value, we have

$$(A - (-1)I) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

To find the nullspace, we row reduce to get

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $\vec{v} = (2, 1)$. We notice that we have not obtained two linear independent eigenvectors, so that we need to look for a generalized eigenvector $\vec{w}$. We have

$$(A - \lambda I)\vec{w} = \vec{v} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

If we set $w_2 = t$, we see that $w_1 = 1 + 2t$ so that we have

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Setting $t = 0$, we have $\vec{w} = (1, 0)$.

It follows that the general solution is given by

$$\vec{x}(t) = \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t}$$

To solve for C_1 and C_2, we utilize the initial conditions $x(0) = -1$ and $y(0) = 2$. At $t = 0$ we have

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which implies

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

so that we have

$$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \end{array}\right]$$

so that $C_1 = 2$ and $C_2 = -5$. It follows that the solution is

$$\begin{aligned} \vec{x}(t) &= \left(2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t} \\ &= \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} - t \begin{bmatrix} 10 \\ 5 \end{bmatrix} \right) e^{-t} \end{aligned}$$

5 Phase Portrait

Now that we have a sense of what the solutions look like, we can construct a much more detailed picture. In fact, we can completely enumerate the possible qualitatively different cases we found when we considered the analytic solutions. We can break things apart something like this (for representative pictures, see Figure 2):

1. **Two real distinct eigenvalues** (or **repeated eigenvalues with two distinct eigenvectors**)
   (a) If both eigenvalues are positive ($\lambda_1 > 0$ and $\lambda_2 > 0$) we say $(0, 0)$ is an *unstable node* or *source*.
   (b) If both eigenvalues are negative ($\lambda_1 < 0$ and $\lambda_2 < 0$) we say $(0, 0)$ is a *stable node* or *sink*.
   (c) If the eigenvalues have opposite sign, we say $(0, 0)$ is a *saddle point*.
2. **Repeated eigenvalue, one eigenvector**
   (a) If the repeated eigenvalue is positive ($\lambda > 0$) we say $(0, 0)$ is a *degenerate source*.
   (b) If the repeated eigenvalue is negative ($\lambda < 0$) we say $(0, 0)$ is a *degenerate sink*.
3. **Complex eigenvalues**
   (a) If the real part of the eigenvalue is positive ($\alpha > 0$) we say $(0, 0)$ is an *unstable spiral* or *source spiral*.
   (b) If the real part of the eigenvalue is negative ($\alpha < 0$) we say $(0, 0)$ is a *stable spiral* or *sink spiral*.
   (c) If the real part of the eigenvalue is zero ($\alpha = 0$) we say $(0, 0)$ is a *center*.
4. **Zero eigenvalue**
   (a) If there is a zero eigenvalue, we say that the system is *degenerate* (there is a line of fixed points through $(0, 0)$).

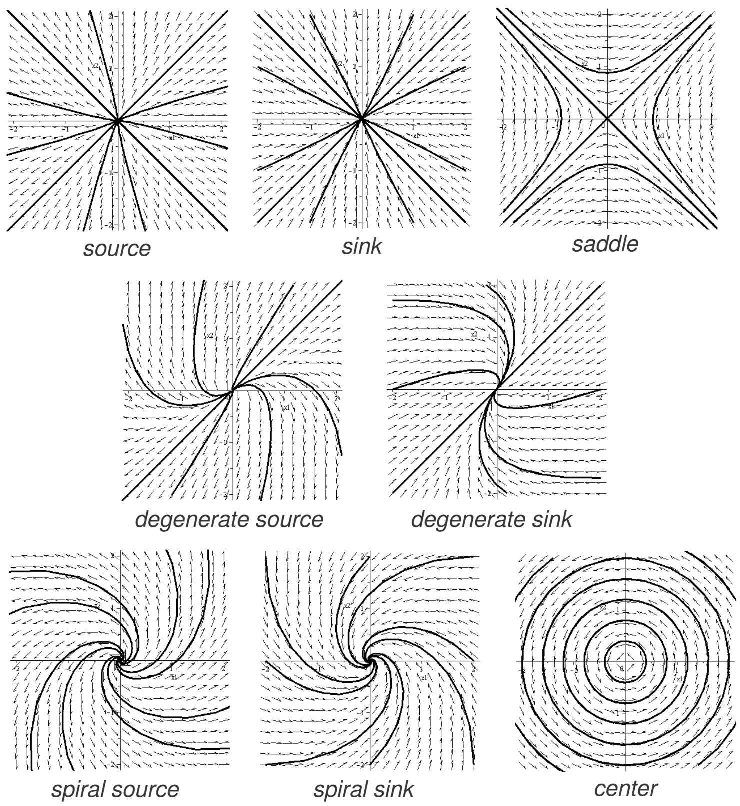

Figure 2: Canonical pictures for the various cases of two-dimensional linear autonomous differential equations.