MATH 521, Spring 2014, Assignment 2

Due date: Wednesday, February 12

Problems for submission:

1. A real number is irrational if it cannot be written in the form a/b, $a, b \in \mathbb{Z}$, $b \neq 0$. The set of irrational numbers is given by $\mathbb{J} = \mathbb{R} \setminus \mathbb{Q}$.

   (a) Prove that, if $p \in \mathbb{Q}$ and $q \in \mathbb{J}$ then $p + q \in \mathbb{J}$ and, if $p \neq 0$, $pq \in \mathbb{J}$.

   (b) Prove that there is an irrational number $r \in \mathbb{J}$ in between any two rational numbers $p, q \in \mathbb{Q}$.

   (c) Prove that the irrational numbers are uncountable. (**Hint:** Show that if A is an uncountable set and B is countably infinite then the set $A \setminus B$ is uncountable.)

   (d) Do the irrational numbers form a field? Either prove the claim that do form a field or derive a contradiction.

2. Prove that if S is a countably infinite set then the power set $\mathcal{P}(S)$ is an uncountable set.

3. Consider an ordered field $\mathbb{F}$ with additive identity $0 \in \mathbb{F}$ and multiplicative identity $1 \in \mathbb{F}$. We define the *absolute value* of $x \in \mathbb{F}$ to be

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

   Prove the following for any $x, y \in \mathbb{F}$:

   (a) $-x = (-1) \cdot x$.

   (b) $x + x = 2 \cdot x$ where $2 = (1 + 1) \in \mathbb{F}$.

   (c) $|x| \geq 0$ with $|x| = 0$ if and only if $x = 0$.

   (d) $|x + y| \leq |x| + |y|$

   (e) $\max(x, y) = \frac{1}{2} \cdot (x + y + |x - y|)$ where $(1/2) = 2^{-1}$

   (f) $\min(x, y) = \frac{1}{2} \cdot (x + y - |x - y|)$ where $(1/2) = 2^{-1}$

4. Suppose that X is an ordered set with the least-upper-bound property and that $S_n \subseteq X$, $n \in \mathbb{N}$, is a family of subsets which are bounded above by a common upper bound. Consider the set

$$S = \{\sup(S_n) \mid n \in \mathbb{N}\}.$$

   (a) Prove that $\sup(S_n) \leq \sup(S)$ for all $n \in \mathbb{N}$.
   (b) Is it possible for the inequality $\sup(S_n) < \sup(S)$ to be *strict* for all $n \in \mathbb{N}$ (i.e. that no supremum of S_n attains the global supremum of S)? Justify your answer.

5. Suppose $S \subseteq X$ where X is an ordered set. Either give an example of an S satisfying the given properties or prove that no such set may exist:

   (a) S is bounded from above and below, and yet neither $\max(S)$ nor $\min(S)$ exist.
   (b) $\min(S)$ and $\max(S)$ exist and $\max(S) = \min(S)$.
   (c) $\min(S)$, $\max(S)$, $\inf(S)$, and $\sup(S)$ exist and $\inf(S) < \min(S) < \max(S) < \sup(S)$.

6. Chapter #1, Question #5 in Rudin.

7. Chapter #1, Question #8 in Rudin.

Honors Question Chapter #1, Question #9 in Rudin.