MATH 521, Spring 2014, Assignment 3

Due date: ~~Wednesday, February 26~~ Monday, March 3

Problems for submission:

1. Prove that the following are metric spaces:

   (a) $(\mathbb{R}^n, d)$ where, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

   $$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y} \\ 1, & \text{otherwise.} \end{cases}$$

   (b) $(\mathbb{R}^n, d_\infty)$ where $d_\infty(\mathbf{x}, \mathbf{y}) = \max\limits_{i=1,\ldots,n} |x_i - y_i|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (c) $(\mathbb{R}^n, \tilde{d})$ where

   $$\tilde{d}(\mathbf{x}, \mathbf{y}) = \frac{d(\mathbf{x}, \mathbf{y})}{1 + d(\mathbf{x}, \mathbf{y})}$$

   for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ where $d(\mathbf{x}, \mathbf{y})$ is any metric on $\mathbb{R}^n$. (**Hint:** Prove first of all that $b \geq a \geq 0$ implies that $\dfrac{b}{1+b} \geq \dfrac{a}{1+a}$!)

2. Consider the three metrics in Question #1 with $\mathbb{R}^2$. Draw or describe the unit ball $B_1(\mathbf{0})$ around the point $\mathbf{0} = (0, 0)$.

3. Consider the *Hausdorff metric*

   $$d(I_1, I_2) = \max\left\{\max_{x \in I_1} \min_{y \in I_2} |x - y|, \max_{y \in I_2} \min_{x \in I_1} |x - y|\right\}$$

   on the space of intervals $I = \{[a, b] \mid a, b \in \mathbb{R}\}$.

   (a) Determine the distance between the following pairs of sets: (i) $I_1 = [0, 1]$ and $I_2 = [1, 2]$, (ii) $I_1 = [0, 1]$ and $I_2 = [-2, 2]$.

   (b) Prove that $d(I_1, I_2)$ satisfies the first two metric properties. (You do not need to prove the Δ-inequality!)

4. Consider the metric space $(\mathbb{R}, d)$ with the metric $d(x, y) = |x - y|$. For the following subsets $S \subseteq \mathbb{R}$, identify the sets S^o, S', ∂S, and $\overline{S}$. Also determine whether the sets S are open, closed, both, or neither.

(a) $S = \{x \in \mathbb{R} \mid |x - y| < r\}$ where $r > 0$ and $y \in \mathbb{R}$ is fixed.

(b) $S = \left\{ \dfrac{1}{n} + \dfrac{1}{m} \in \mathbb{R} \mid n, m \in \mathbb{N} \right\}$.

(c) $S = \{x \in \mathbb{R} \mid 0 < x < 1, x \text{ is irrational}\}$.

5. Suppose (X, d) is a metric space and $S \subseteq X$. Prove the following:

   (a) S' is closed.

   (b) $\overline{S} = S \cup \partial S$.

   (c) $\overline{S^c} = (S^o)^c$

   (d) $\partial S = \overline{S} \cap \overline{S^c}$

6. Suppose (X, d) is a metric space and $A, B \subseteq X$. Prove the following:

   (a) $(A \cap B)^o = A^o \cap B^o$

   (b) $(A \cup B)^o \supseteq A^o \cup B^o$

   (c) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

   (d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (e) Find sets $A, B \subseteq \mathbb{R}$ such that the inclusions (b) and (c) are strict.

Honors Question #1 Consider a metric space (X, d) equipped with the discrete metric $d(x, y) = 0$ for $x = y$ and $d(x, y) = 1$ for $x \neq y$.

(a) What are the open subsets $S \subseteq X$ in this topology? What are the closed sets? Justify your claims.

(b) ~~Is X connected? Why or why not.~~

Honors Question #2 Is it true that $S^o = (\overline{S})^o$? Either prove the claim that it is true or find a counter-example.