MATH 521, Spring 2014, Assignment 4

Due date: Friday, March 14

Problems for submission:

1. Suppose (X, d) is a metric space. Prove that the following statements are equivalent:
   (a) X is connected;
   (b) the only subsets $S \subseteq X$ which are both open and closed are $S = X$ and $S = \emptyset$; and
   (c) ∂S is nonempty for every $S \subseteq X$ other than $S = X$ and $S = \emptyset$.

2. Consider the metric space $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$. We will say that $I \subseteq \mathbb{R}$ is an *interval* if I is nonempty and, for every $x, y \in I$, and every z such that $x < z < y$, we have that $z \in I$.
   (a) Prove that the only connected subsets of $\mathbb{R}$ are intervals. (Note: Prove that intervals are connected as part of the argument!)
   (b) Prove that if $A, B \subseteq \mathbb{R}$ are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.
   (c) Prove that if $S \subseteq \mathbb{R}$ is connected then S^o is connected.
   (d) Show that (b) and (c) do not hold if we consider $(\mathbb{R}^2, d_2)$ where d_2 is the standard Euclidean metric.

3. Consider the metric space $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$. Show that the following sets are not compact by finding an open cover which does not have a finite subcover.
   (a) $A = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$
   (b) $B = \left\{\frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N}\right\}$
   (c) $C = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$

4. Prove that the set
$$S = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}$$
is compact directly from the definition (i.e. do not use the Heine-Borel Theorem).

5. Suppose (X, d) is a metric space. Prove the following:

   (a) If $\{S_n\}_{n=1,\ldots,N}$ is a family of compact sets such that $S_n \subseteq X$ for all $n = 1, \ldots, N$, then $\bigcup_{n=1}^{N} S_n$ is compact.

   (b) If $\{S_\alpha\}_{\alpha \in A}$ is a family of compact sets such that $S_n \subseteq X$ for all $\alpha \in A$, then $\bigcap_{\alpha \in A} S_\alpha$ is compact. (**Hint:** You may use the fact that compact sets are closed and that an arbitrary intersection of closed sets is closed. Also note this intersection may be empty.)

6. Let $S \subseteq X$ be a closed subset of a metric space (X, d). A family of sets $\{S_\alpha\}$ such that $S_\alpha \subseteq S$ for all $\alpha \in A$ is said to have the *finite intersection property* if every finite subfamily $\{S_{\alpha_i}\}$, $i \in \{1, \ldots, N\}$, of $\{S_\alpha\}$ has the property

$$\bigcap_{i \in \{1,\ldots,N\}} S_{\alpha_i} \neq \emptyset.$$

   Suppose that every family $\{S_\alpha\}$, $\alpha \in A$, of *closed* subsets of S which has the finite intersection property also satisfies

$$\bigcap_{\alpha \in A} S_\alpha \neq \emptyset.$$

   Prove that S is compact.

Honors Question #1 Suppose (X, d) is a metric space and $S \subseteq X$ is a connected subset with more than one element. Show the following:

   (a) Every point of S is a limit point.

   (b) S is uncountable.