# MATH 521, Spring 2014, Assignment 5

Due date: Wednesday, April 2

**Problems for submission:**

1. Use the definition of the limit to prove that the sequences $\{a_n\}$ with the given terms converge to the indicated limits:

   (a) $a_n = \dfrac{n^2}{2n^2 + n - 1}$, $L = 1/2$

   (b) $a_n = \sqrt{n^2 + n} - n$, $L = 1/2$

2. Consider a sequence $\{a_n\} \subset \mathbb{R}$ which is recursively defined by

$$a_{n+1} = f(a_n). \tag{1}$$

   (a) Prove that if there is some $L \in \mathbb{R}$ and a $0 \leq c < 1$ such that

$$\left| \frac{a_{n+1} - L}{a_n - L} \right| < c$$

   for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} a_n = L$.

   (b) Consider the sequence (1) with $f(a_n) = \dfrac{1 - 2a_n}{3}$. Use the result of part (a) to show that $\{a_n\}$ converges to the limit $L = 1/5$ regardless of the value of $a_1$. What is the value of $c$?

   (c) Consider the sequence (1) with $f(a_n) = 1 + \dfrac{1}{1 + a_n}$ and $a_1 = 1$. Use the result of part (a) to show that $\{a_n\}$ converges to the limit $L = \sqrt{2}$. What is the value of $c$? (Hint: first show that $1 \leq a_n \leq 2$ implies $1 \leq a_{n+1} \leq 2$, or some comparable bound.)

3. Consider the sequence $\{a_n\}$ where $a_{n+2} = a_{n+1} + a_n$, $n \in \mathbb{N}$.

   (a) Consider the initial values $a_1 = 1$ and $a_2 = 1$. Explicitly determine up to the term $a_6$ and then prove that the sequence diverges. (Hint: Show it is monotonically increasing but has no upper bound.)

(b) Consider the initial values $a_1 = 1$ and $a_2 = \frac{1-\sqrt{5}}{2}$. Explicitly determine up to the term $a_6$. Then prove inductively that

$$a_{n+1} = \left(\frac{1-\sqrt{5}}{2}\right) a_n$$

and use this to prove that the sequence converges to the limit $L = 0$.

4. Chapter #3, Question #20 in Rudin

5. Determine the limit inferior and limit superior of the following sequences (you do not need to rigorously prove the limits, but the limit values need to be exact):

$$\text{(a) } a_n = (-1)^n \left(\frac{n+1}{n-1}\right) \arctan(n), \quad \text{(b) } a_n = \sin\left((2n-1)\frac{\pi}{2} + \frac{1}{n}\right)$$

$$\text{(c) } a_{2n} = -\frac{1}{2} a_{2n-1},\ a_{2n+1} = a_{2n} + 1,\ a_1 = 0.$$

6. Consider a sequence of real numbers $\{a_n\}$ which is bounded. Define the set $S$ to be the set of all subsequential limits of $\{a_n\}$.

   (a) Prove that $s \in S$ if and only if, for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there is an $n > N$ so that $|a_n - s| < \epsilon$.

   (b) Define $S^* = \sup(S)$. Prove that $\alpha \in \mathbb{R}$ satisfies the following two conditions if and only if $\alpha = S^*$:

      (i) For every $\epsilon > 0$, there exists an $n \in \mathbb{N}$ so that $m > n$ implies that $a_m < \alpha + \epsilon$;

      (ii) For every $\epsilon > 0$ and every $n \in \mathbb{N}$, there exists an $m > n$ so that $a_m > \alpha - \epsilon$.

   (c) **Honors only!** Define the limit superior of $\{a_n\}$ as in class:

   $$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{m>n} a_n\right).$$

   Prove that $S^* = \limsup_{n\to\infty} a_n$. (Hint: Use part (b)!)

**Honors Question** Chapter #3, Question #23 in Rudin