MATH 521, Spring 2014, Assignment 6

Due date: Friday, April 18

Problems for submission:

1. Consider series $\sum_{n=1}^{\infty} a_n$, where $a_n \in \mathbb{R}$.

   (a) Prove that the series with terms $a_n = \sqrt{n+1} - \sqrt{n}$ diverges.

   (b) Prove that the series with terms $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$ converges.

2. Use the definition of continuity to prove that $f(x) = x^n$ ($n \in \mathbb{N}$) is continuous on $\mathbb{R}$ in the metric space $(\mathbb{R}, d)$ with $d(x, y) = |x-y|$. [Hint: Use induction!]

3. Consider $f(x) = \sqrt{x}$ in the metric space (X, d) with $X = [0, \infty)$ and $d(x, y) = |x - y|$.

   (a) Prove that $f(x)$ is uniformly continuous in X. (Hint: Consider first bounding $|f(x) - f(y)|^2$ by $\delta = \epsilon^2$!)

   (b) Suppose that $\epsilon = 0.25$. What is the largest possible δ value which can be chosen uniformly for all $x, y \in X$? Explain your reasoning. (Hint: Consider the graph!)

4. Consider a function $f : \mathbb{R} \mapsto \mathbb{R}$ which is continuous on all of $\mathbb{R}$. Find examples satisfying the following:

   (a) $A \subseteq \mathbb{R}$ is open but $f(A)$ is not.

   (b) $B \subseteq \mathbb{R}$ is closed but $f(B)$ is not.

   (c) $C \subseteq \mathbb{R}$ is connected but $f^{-1}(C)$ is not.

   (d) $D \subseteq \mathbb{R}$ is compact but $f^{-1}(D)$ is not.

5. Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \mapsto Y$ is uniformly continuous. Prove that, if $\{x_n\}$ is a Cauchy sequence in (X, d_X) then $\{f(x_n)\}$ is a Cauchy sequence in (Y, d_Y). Does this result hold if f is only continuous? Why or why not?

6. Consider a metric space (X, d). Let $f : X \mapsto X$ be a function which satisfies

$$d(f(x), f(y)) \leq cd(x, y)$$

for all $x, y \in X$, where $c \in [0, 1)$. (Such a function is called a *contraction mapping*.) For any $x \in X$, define the sequence $\{x_n\}$ with terms

$$x_{n+1} = f(x_n), \quad x_1 = x.$$

Note that $x_{n+1} = f^n(x)$ where $f^n(x) = \underbrace{f(f(\cdots(f(x))}_{\text{n times}}$.

(a) Prove that every contraction mapping $f : X \mapsto X$ is uniformly continuous on X.

(b) Prove that, for all $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leq c^{n-1} d(x, f(x)).$$

(c) Prove that, for all $n, m \in \mathbb{N}$, such that $m > n$, we have

$$d(x_n, x_m) \leq \frac{c^{n-1}}{1-c} d(x, f(x)).$$

(Hint: use part (b) and consider the triangle inequality and the geometric series.)

(d) Prove that $\{x_n\}$ is a Cauchy sequence. (Hint: use part (c)!)

(e) Suppose $S \subseteq X$ is compact and $f : S \mapsto S$. Prove that there is a point $x^* \in S$ such that $f(x^*) = x^*$. Also prove that this point is unique. (Hint: use part (a) and (d), but note that we have only proved so far that the sequence is *Cauchy*, not convergent.)

Honors Question #1 Chapter #3, Question #7 in Rudin [Hint: Use the Cauchy criterion and note that, for $\epsilon > 0$, we may eventually bound

$$\sqrt{\sum_{k=n+1}^{m} a_k} \quad \text{and} \quad \sqrt{\sum_{k=n+1}^{m} \frac{1}{k^2}}$$

by $\sqrt{\epsilon}$. The Cauchy-Schwarz Inequality, carefully applied, helps.]

Honors Question #2 Chapter #4, Question #2 in Rudin