MATH 521, Spring 2014, Assignment 8

Due date: Monday, May 5

Submit only the starred (*) questions! (Honors students submit the Honors Question as well.)

1.* Suppose $f : [a, b] \mapsto \mathbb{R}$ is bounded on $[a, b]$. Prove that if f is Riemann integrable on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\, dt$$

is uniformly continuous on $[a, b]$.

2. Suppose $f : [0, 2] \mapsto \mathbb{R}$ is continuous on $[0, 2]$, and that $\int_0^2 f(t)\, dt = 2$. Show that there is a $c \in (0, 2)$ such that $f(c) = c$. [Hint: Consider the function $F(x) = \int_0^x (f(t) - t)\, dt$ and apply the FTC.)

3.* Use the definition of Riemann integrability, or an equivalent formulation, to prove that

$$\int_a^b x\, dx = \frac{1}{2}\left(b^2 - a^2\right).$$

4. Suppose that $f, g : [a, b] \mapsto \mathbb{R}$ are Riemann integrable on $[a, b]$. Define the following:

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2\, dx}$$

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

(There are the basis for the L^2 metric space on *functions*. Note that the objects in this space are functions, rather than vectors as they were in $\mathbb{R}^n$. Nevertheless, we can see immediate parallels with the Euclidean norm $\| \cdot \|_2$ and dot product which is often written in inner product notation $\langle \cdot, \cdot \rangle$.)

(a) Prove directly that, if $f(x)$ is Riemann integrable on $[a, b]$, then $f(x)^2$ is Riemann integrable on $[a, b]$. (This guarantees that we may evaluate $\|f\|_2$.)

(b)* Compute $\|f_n\|_2$, $n \in \mathbb{N}$, for the family of functions

$$f_n(x) = \begin{cases} 1 + nx, & \text{for } -1/n < x \leq 0 \\ 1 - nx, & \text{for } 0 < x \leq 1/n \\ 0, & \text{otherwise} \end{cases}$$

on the interval $[-1, 1]$. What happens as $n \to \infty$? Does this make sense in the context of the individual functions $f_n(x)$? Are there are peculiarities? (Note: The norm is a measure of "distance" of a function from the trivial "zero function" $f(x) = 0$.)

(c)* Prove that $|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2$. Hint: It will help to look back at the proof of the Cauchy-Schwarz Inequality contained in the Week 4 notes. Consider starting with the identity

$$\int_a^b \int_a^b (f(x)g(y) - f(y)g(x))^2 \, dy \, dx \geq 0.$$

(d) Prove that $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$. (Hint: use (c)!)

Honors! Define the set $L_2([a, b])$ to be the set of all functions which are Riemann integrable on $[a, b]$. Define $d : L_2([a, b]) \times L_2([a, b]) \mapsto \mathbb{R}$ according to

$$d(f, g) = \|f - g\|_2.$$

Show that $d(f, g)$ satisfies all the metric space condition *except* $d(f, g) = 0$ implies $f(x) = g(x)$. (There are subtleties with this final metric space condition which require some degree of measure theory to resolve.)