MATH 521, WEEK 1:
Introduction, Definitions & Review

1 Introduction

The purpose of this course is to give a rigorous introduction to the subject of **mathematical analysis**.

So what distinguishes mathematical analysis from the mathematics courses you are probably already familiar with (and hopefully enjoy!) such as calculus, linear algebra, and geometry? The first (and primary) distinction is an emphasis on **axiomatic definitions** and **rigorous proofs**. There is no way around it at this point: this course will be **proof-intensive**. In fact, we will probably see at least one proof in every single lecture! This leads to a few very important notes before we begin:

1. **Every proof is unique**. We will not generally be able to simply identify types of problems and apply a set formula for solving them as we have in previous courses (e.g. recognizing an integral can be solved by substitution, or a differential equations is separable, etc.). We will often have to be very creative and use non-intuitive methods!

2. **Details will be important.** It will no longer be enough to justify things because they are familiar or "well-understood" or because they work for a particular example we happen to like. We will be interested in asking the question of precisely what underlying properties allow our claims to follow.

3. **Proofs must follow logically.** We will have to be careful to ensure our arguments are consistent and actually prove what we have set out to prove. It is important at the outset, therefore, to go over a few basic motifs in propositional logic. In particular, we will need to distinguish between **direct** and **indirect** proof methods, and (re-)familiarize ourselves with common proof methods such as the **method of induction** and **proof by contradiction**.

A second characterizing feature of mathematical analysis is that we will introduce a general notion of distance between objects (called a **metric**). This distinguishes analysis from other areas of mathematics such as abstract algebra, number theory, and group/ring theory (with which you may not be as familiar). This notion will allow us to prove many results about the **existence**, **uniqueness**, and **convergence** of a wide variety of mathematical objects. In addition to the conventional distance between points in space (i.e. Euclidean distance), we will also consider distances between *functions*, and even distances between *sets*. The key realization we will make is that, once we have the bedrock theory of metric spaces understood, we will be able to treat all of these very distinct objects as though they were the same!

Of course, we will not be able to cover everything under the umbrella of mathematical analysis in this course, but we should be able to get a good start. The tentative plan is to cover the following topics (roughly in the order they are presented in Rudin):

1. Number Systems and Set Theory
    - Ordered Sets, Fields, Integers and Rational Numbers, Real and Complex Numbers, Euclidean Spaces, Countable and Uncountable Sets, etc.
2. Metric Spaces and Basic Topology
    - Metrics, Open and Closed Sets, Closure, Open Covers, Compact Sets, Connected Sets, Dense Sets, etc.
3. Sequences and Series
    - Limits, Convergence, Subsequences, Cauchy Sequences, $\limsup$ and $\liminf$, Root and Ratio Tests, etc.
4. Functions and Continuity
    - Definitions, Continuity and Compactness, Discontinuities, etc.
5. Differentiation
    - Formal Definition, Continuity of Derivatives, L'Hopital's Rules, Higher-Order Derivatives, etc.
6. Integration
    - Partitions, Riemann-Stieltjes Integral, Definite Integrals, Fundamental Theorem of Calculus, etc.
7. Sequences and Series of Functions

- Uniform and Pointwise Convergence, Equicontinuity, Arzela-Ascoli Theorem, etc.

Before we jump into the material, it will be important to quickly cover the expectations going into the course, the notation we will use, and give a quick tutorial on valid methods of mathematical proof.

1.1 Expectations and Review

Mathematical analysis is, by its nature, self-contained and therefore is, in principle, accessible to everybody regardless of background. That said, it will be tremendously beneficial to have some formal background in **calculus**, **set theory**, and **formal logic** prior to engaging in analysis. We will quickly review some pertinent elements from these topics now.

1.1.1 Number Systems

Different applications require the use of different notions of what a "number" is. For instance:

1. If we are counting apples from a harvest, or beads on a necklace, or cards in a deck, we are interested in whole numbers like 1, 2, 3, etc. (and maybe 0).

2. If we are determining the probability of a particular hand in poker, we are interested in the ratio of the number of hands of interest (an integer value) relative to the number of overall possible hands (an integer value).

3. If we are measuring the length of a wall or the area of a room (in ft or ft^2, m or m^2) we are interested in the most accurate decimal value we can obtain.

4. If we are interested in finding all of the roots of a quadratic (i.e. solving an expression of the form $ax^2+bx+c=0$), we must allow for numbers with $i=\sqrt{-1}$.

These are cases where we are interested in the *natural*, *rational*, *real*, and *complex* numbers respectively. We should be familiar with these numbers systems from previous courses. At any rate, we will take the following notation as given throughout the course:

$\mathbb{N}$	**natural** or **counting numbers**, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{Z}$	**integers**, i.e. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	**rational numbers**, i.e. numbers of form a/b, $a, b \in \mathbb{Z}, b \neq 0$
$\mathbb{R}$	**real numbers**, i.e. any (potentially non-terminating) decimal number
$\mathbb{J}$	**irrational numbers**, i.e. real numbers not of form a/b, $a, b \in \mathbb{Z}, b \neq 0$
$\mathbb{C}$	**complex numbers**, i.e. any $z = a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

1.1.2 Quantifiers

Throughout this course, we will frequently encounter statements of the form "For every..." and "There exists a...". We will also frequently have to *negate* these statements. That is to say we will come across statements like "It is *not true* that for every..." and "There does *not* exist a...".

We should already have been exposed to several such instances in the course of our calculus education. For instance, while we understand continuity of a function $f(x)$ *informally* as meaning that the function $f(x)$ has no "breaks" (i.e. we can sketch it without taking our pencil off the paper), the *formal* definition is:

$$\begin{array}{c}\text{for every } \epsilon > 0 \text{ there is a } \delta > 0 \text{ so that} \\ |x - a| < \delta \text{ implies } |f(x) - f(a)| < \epsilon.\end{array} \tag{1}$$

We can work with such a statement, but it is certainly a mouthful. It is common in analysis (and proposition logic, in general) to introduce the following short-hand notation:

$\forall$	"for every", "for all"
$\exists$	"there exists", "there is"
$\exists!$	"there exists a unique", "there is exactly one"
s.t.	"so that", "such that"
$\neg$	"negation", "it is not true that"
$\implies$	"implies", "therefore"
'iff', $\iff$	"if and only if"

We can now rewrite the definition of continuity of $f(x)$ at a point a in the condensed form

$$\forall\, \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \forall\, x \in \mathbb{R}, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \tag{2}$$

In this course, we will consider either the expanded or condensed forms (1) or (2) to be equally valid (it is the meaning and logical implications of the statements in which we will be interested!). It is worth familiarizing ourselves with condensed forms like (2) because they distill the logical connections between the objects in the statement. In particular, it becomes formulaic to *negate* such statements. Supposing $P(x)$ and $Q(x)$ are statements whose truth value depends on $x \in S$, we have the following results

$$\begin{aligned} \neg(\neg P(x)) &\implies P(x) \\ \neg(P(x) \text{ and } Q(x)) &\implies \neg P(x) \text{ or } \neg Q(x) \\ \neg(P(x) \text{ or } Q(x)) &\implies \neg P(x) \text{ and } \neg Q(x) \\ \neg(P(x) \implies Q(x)) &\implies P(x) \text{ and } \neg Q(x) \\ \neg(\forall\, x \in S,\ P(x)) &\implies \exists\, x \in S \text{ s.t. } \neg P(x), \\ \neg(\exists\, x \in S \text{ s.t. } P(x)) &\implies \forall\, x \in S,\ \neg P(x) \end{aligned} \tag{3}$$

Let's practice manipulating expressions with quantifiers by manipulating a few simple examples.

Example 1: Restate the following expression using quantifier notation, and then determine the *negation* of the statement:

Every integer is larger than some other integer.

Proof: This statement is clearly *true*, but we are not concerned with that detail right (we will spend plenty of time later in this course proving things!). We would like to simply state this using our accepted notation, in order to distill how exactly everything relates. After a little thought, we should accept that the statement we need is

$$\forall\, x \in \mathbb{Z}, \exists\, y \in \mathbb{Z} \text{ s.t. } x > y.$$

Now consider the second question of negating this statement. We apply the rules (3) to get

$$\begin{aligned} &\neg(\forall\, x \in \mathbb{Z}, \exists\, y \in \mathbb{Z} \text{ s.t. } x > y) \\ &\exists\, x \in \mathbb{Z} \text{ s.t. } \neg(\exists\, y \in \mathbb{Z} \text{ s.t. } x > y) \\ &\exists\, x \in \mathbb{Z} \text{ s.t. } \forall\, y \in \mathbb{Z},\ \neg(x > y) \\ &\exists\, x \in \mathbb{Z} \text{ s.t. } \forall\, y \in \mathbb{Z},\ y \geq x. \end{aligned}$$

After a little simplifying, we can read this as "There is an integer which is smaller than or equal to every other integer." This is clearly *false* but, again, that is not our concern (yet!). All we are concerned with is that this is the correct opposite statement to the original one. $\square$

Example 2: State the *negation* of (2).

Solution: Since (2) corresponds to the definition of continuity at a point a, what we are really being asked for is the definition of *dis*continuity at a point a. We want to use the definition given to derive a formal definition of discontinuity.

Using (3), we have that the negation is

$$\begin{aligned}
&\neg(\forall\, \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \forall\, x \in \mathbb{R}, |x-a| < \delta \implies |f(x) - f(a)| < \epsilon) \\
\implies &\exists\, \epsilon > 0 \text{ s.t. } \neg(\exists\, \delta > 0 \text{ s.t. } \forall\, x \in \mathbb{R}, |x-a| < \delta \implies |f(x) - f(a)| < \epsilon) \\
\implies &\exists\, \epsilon > 0 \text{ s.t. } \forall\, \delta > 0, \neg(\forall\, x \in \mathbb{R}, |x-a| < \delta \implies |f(x) - f(a)| < \epsilon) \\
\implies &\exists\, \epsilon > 0 \text{ s.t. } \forall\, \delta > 0, \exists\, x \in \mathbb{R} \text{ s.t. } \neg(|x-a| < \delta \implies |f(x) - f(a)| < \epsilon) \\
\implies &\exists\, \epsilon > 0 \text{ s.t. } \forall\, \delta > 0, \exists\, x \in \mathbb{R} \text{ s.t. } |x-a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon.
\end{aligned}$$

In other words, in order for there to be a *discontinuity* at a point $a \in \mathbb{R}$, it is necessary that

There exists a $\epsilon > 0$ such that for every $\delta > 0$ there is an $x \in \mathbb{R}$ so that $|x-a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$.

This is certainly a mouthful, but it is a necessary mouthful. If we are dealing with discontinuities, this is what we are dealing with! $\square$

1.1.3 Set Theory

It will be important throughout this course to be able to read and write using **set notation**. For instance, we should be able to quickly understand that

$$S = \{x \in \mathbb{R} \mid x^2 - x - 2 = 0\}$$

means that S is the set of all real numbers satisfying the equation $x^2 - x - 2 = 0$. This is an example of an axiomatically defined set, since the values of x are defined by axioms. Notice, however, that since there are only two values which satisfy the given equation we could also write

$$S = \{-2, 1\}.$$

This is an example of an *enumerated* set—i.e. a set where all the elements are explicitly stated. We should be comfortable using such notation, and recognizing the equivalence of the two formulations for S given above.

We should re-familiarize ourselves with the following definitions:

Definition 1.1. *Suppose A and B are subsets of a universal set X. Then:*

1. $A \subseteq B$ *if* $x \in A$ *implies* $x \in B$*;*
2. $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$*;*
3. $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$*;*
4. $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$*;*
5. $A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}$*; and*
6. $A^c = \{x \in X \mid x \notin A\}$.

We will often have to prove things in this course using set theoretic notation. Let's practice now!

Example 1: Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof: In the past, we may have been able to get away with proving this by appealling to Venn diagrams. In fact, that would be valid for this particular problem, since it simple enough that such a diagram captures all the pertinent information. We should practice, however, proving such statements formally.

We have

$$\begin{aligned} & x \in (A \cup B)^c \\ (1) \quad \Longrightarrow \; & x \notin (A \cup B) \\ (2) \quad \Longrightarrow \; & x \notin A \text{ and } x \notin B \\ \Longrightarrow \; & x \in A^c \text{ and } x \in B^c \\ \Longrightarrow \; & x \in (A^c \cap B^c). \end{aligned}$$

It follows that $(A \cup B)^c \subseteq A^c \cap B^c$. The inclusion $A^c \cap B^c \subseteq (A \cup B)^c$ follows from taking the above argument in the reverse direction. It follows that $(A \cup B)^c = A^c \cap B^c$.

The only non-trivial step in the above reasoning is between (1) and (2). Notice that we have $A \cup B = \{y \in X \mid y \in A \text{ or } y \in B\}$. It is clear that if $x \in A$ then $x \in A \cup B$ and if $x \in B$ then $x \in A \cup B$ so that we must have $x \notin A$ *and* $x \notin B$. $\square$

Example 2: Prove that $A \cap B = A \setminus (A \setminus B)$.

Proof: We need to prove $A \cap B \subseteq A \setminus (A \setminus B)$ and $A \setminus (A \setminus B) \subseteq A \cap B$. First take $x \in A \setminus (A \setminus B)$. It follows that

$$\begin{aligned} &\implies x \in A \text{ and } x \notin A \setminus B \\ &\implies x \in A \text{ and } x \in (A \setminus B)^c. \end{aligned} \tag{4}$$

Notice that $y \in A \setminus B$ implies that $y \in A$ and $y \notin B$ so that $x \in (A \setminus B)^c$ implies $x \notin A$ or $x \in B$. It follows that we have

$$\implies \quad x \in A \text{ and } (x \notin A \text{ or } x \in B). \tag{5}$$

We know that $x \in A$ so that the only way ($x \notin A$ or $x \in B$) can be true is if $x \in B$. It follows that we have

$$\begin{aligned} &\implies \quad x \in A \text{ and } x \in B \\ &\implies \quad x \in A \cap B \end{aligned} \tag{6}$$

so that $A \setminus (A \setminus B) \subseteq A \cap B$.

Now suppose $x \in A \cap B$. We can easily reverse the order of proof to show that (6), then (5), then (4) hold so that $A \cap B \subseteq A \setminus (A \setminus B)$. This implies $A \cap B = A \setminus (A \setminus B)$ and we are done. □

1.1.4 Proof Methods

Given the proof-intensive nature of this course, it is important to make sure at the outset that we understand what constitutes a valid mathematical proof, and what the common methods of proof are.

We distinguish between **direct** and **indirect** proof methods. Throughout what follows we will consider P to be the set of premises of an argument and Q to be the set of conclusions.

1. **Direct Proof**: A proof in which the claim is shown to logically follow from the assumptions. The key feature is the starting point is to assume that the premises are *true*. General form of the argument is assuming P and showing $P \implies Q$.

2. **Indirect Proof**: A proof in which the negation of the claim is shown to lead to an absurdity. The key feature is that the starting point is to assume the claim is *false*. General form of the argument is assuming $\neg Q$ and showing $\neg Q \implies \neg P$. (This is also called *reductio ad absurdum* or *proof by contradiction*.)

We make the following notes:

- The logical forms $P \implies Q$ and $\neg Q \implies \neg P$ are *logically equivalent.* The latter form is called the **contrapositive** and is used frequently in mathematics and propositional logic. (We can build some intuition about why this formula works by taking true statements and constructing contrapositives! e.g. "Every bear is a mammal" becomes "Every non-mammal is not a bear", "Billionaires are wealthy" becomes "Poor people are not billionaires", etc.)
- It is important not to confuse the contrapositive form $\neg Q \implies \neg P$ with the **converse** $Q \implies P$. It is *not true* in general that $P \implies Q$ implies $Q \implies P$, although sometimes it is true as well. (Try this with the previous examples! It is clearly not true that "Every mammal is a bear" or that "Being wealthy implies you are a billionaire", etc.)

Example: Prove that the square of any odd number is odd.

Direct proof: The first step is to re-state the problem mathematically. In this case, we have

$$n \text{ is odd} \quad \implies \quad n^2 \text{ is odd.}$$

Using the above notation, we have $P =$"n is odd" and $Q =$"n^2 is odd". We apply the direct proof method of assuming that n is odd in order to prove the claim. It follows that we may write $n = 2k+1$ for some $k \in \mathbb{Z}$. It follows directly that

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k+2) + 1 = 2m + 1$$

where $m \in \mathbb{Z}$ because $2k + 2 \in \mathbb{Z}$. We have succeeded in showing directly that n being odd implies n^2 is odd, so that we are done. (Notice that, since every integer is either even or odd, this result also implies that if n^2 is even then n is even by the contrapositive form $\neg Q \implies \neg P$ where $\neg P =$"n is even" and $\neg Q =$"n^2 is even".) $\square$

Example: Show that $\sqrt{2}$ is an irrational number.

Indirect proof: We would justifiably feel adrift if were asked to prove this in a *direct* manner. After all, irrational numbers are defined only by what they are *not*—namely, rational numbers. But therein lies the key: we will assume the negation in the hopes of arriving at a contradiction.

We assume that $\sqrt{2}$ *is* a rational number so that

$$\sqrt{2} = \frac{m}{n}$$

for some $m, n \in \mathbb{Z}$. We may furthermore assume m and n may be selected so that *at least one* of them is odd. We are allowed to assume this by noting that, if both of them were even we would be able to divide by 2 to get new numbers m and n. (If we want to tighten this assumption, we could assume that n and m have no common factors, which we are clearly allowed to do. We will not, however, need the full power of such an assumption.)

It follows that

$$\sqrt{2}n = m \quad \Longrightarrow \quad 2n^2 = m^2.$$

It follows that m^2 is an even number. It follows by our previous result that m is even. It follows that $m = 2k$ for some $k \in \mathbb{Z}$. Substituting this into the equation yields

$$2n^2 = (2k)^2 \quad \Longrightarrow \quad 2n^2 = 4k^2 \quad \Longrightarrow \quad n^2 = 2k^2.$$

It follows now that n^2 is even so that n is even. However, this violates our assumption that at least one of m or n is odd. Since we have reached a contradiction, it must be the case that our assumption (that $\sqrt{2}$ is a rational number) was made in error. It follows that $\sqrt{2}$ is an irrational number, and we are done. $\square$

A particularly common direct proof method, which has a slightly different appearance, is the **inductive proof method**. What gives this method its distinct flavor is the element of **recursion**: we will prove that each step implies the next.

Example: Prove that the sequence $\{a_1, a_2, a_3, \ldots\}$ defined by

$$a_1 = -1, \quad a_{n+1} = 1 + \sqrt{1 + a_n}, \; n \geq 1$$

has the properties (a) $a_n < 3$ for all $n \geq 1$ and (b) $a_{n+1} > a_n$ for all $n \geq 1$.

Inductive Proof: Notice that we can directly verify the required relationships for the first few terms in the sequence. We have that

$$a_1 = -1, a_2 = 1, a_3 \approx 2.4142, a_4 \approx 2.8478, \ldots.$$

This, however, is *insufficient* to prove the result, since the claim holds for an infinite number of terms (all a_n, $n \geq 1$). Even if we spent the rest of the week performing this process, we would *still* have an infinite number of terms to check! So what can we do?

The trick is to *assume* the result holds and then prove that it holding for any individual term a_n (or pair of terms a_n/a_{n+1}) implies it holds for the next term a_{n+1} (respectively, next pair of terms a_{n+1}/a_{n+2}). We have the following reasoning:

Objective (a): Show $a_1 = -1$, $a_{n+1} = 1 + \sqrt{1 + a_n}$, $n \geq 1$, implies $a_n < 3$ for all $n \geq 1$.

Base case: Clearly $a_1 = -1 < 3$.

Inductive case: Assume $a_n < 3$. This implies that we have

$$1 + a_n < 4 \implies \sqrt{1 + a_n} < 2 \implies 1 + \sqrt{1 + a_n} < 3.$$

That is to say, if $a_n < 3$, then $a_{n+1} < 3$. Since $a_1 = -1 < 3$, it follows that $a_n < 3$ for all $n \geq 1$.

Objective (b): Show $a_1 = -1$, $a_{n+1} = 1 + \sqrt{1 + a_n}$, $n \geq 0$, implies $a_{n+1} > a_n$ for all $n \geq 0$.

Base case: Clearly $a_1 = -1$ and $a_2 = 1$ satisfies $a_2 > a_1$.

Inductive case: Assume $a_{n+1} > a_n \geq -1$. This implies that we have

$$1 + a_{n+1} > 1 + a_n \geq 0 \implies \sqrt{1 + a_{n+1}} > \sqrt{1 + a_n} \geq 0$$

$$\implies 1 + \sqrt{1 + a_{n+1}} > 1 + \sqrt{1 + a_n} \geq 1 > -1.$$

(Note that the ≥ -1 was only introduced to ensure that the root could taken.) It follows that $a_{n+2} > a_{n+1} \geq -1$. Since $a_2 > a_1 \geq -1$, it follows that $a_{n+1} > a_n$ for all $n \geq 0$. □

Note: The base case is subtle but important. For instance, we can easily check that $a_1 = 4$ implies $a_2 = 3.2361$ so that $a_2 < a_1$. This seems like a contradiction! In fact, for any sequence starting at $a_1 > 3$ we have $a_2 < a_1$ (and $a_{n+1} < a_n$ for $n \geq 1$ thereafter). But we have already proved a result stated basically the opposite. What gives?

The reason this is not a true contradiction is subtle, but present. We require in the inductive step that $a_2 < a_1$. If this is violated—as it clearly is in this case!—we may not conclude that $a_{n+1} < a_n$ for $n \geq 0$.

Note: We will revisit examples like this when we consider sequences and subsequences in a few weeks. This sequences, and sequences like it, will allow us to form some intuition on the relationship between upper and lower bounds, monotonicity, and convergence.