MATH 521, WEEK 6:
Open and Closed Sets, Closure, Connected Sets

1 Open and Closed Sets

We should feel happy with what we have accomplished so far. We have identitied a formal, axiomatic way to capture the idea of "distance" between two points in a set. Importantly, our axioms are general enough to be applicable to a wide variety of settings (although we will predominantly consider Euclidean spaces $\mathbb{R}^n$).

So what can we do with this newly-minted "metric"? Returning to our map example, if we are heading out for dinner, the first thing we would probably want to do is characterize which restaurants are "nearby", i.e. in our neighborhood. We might be interested in travelling five miles, but not ten. So we might be interested in the set of all restaurants within five miles. This should give us a better sense of what the terrain (i.e. *topology*, in math-speak) really looks like.

We have the following definitions.

Definition 1.1. *Let (X, d) be a metric space. Take $x \in X$ and $r > 0$. We define the* ***open ball*** *(or simply* ***ball****) of radius r centered at x to be the set*

$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$

A ball centered at $x \in X$ is said to be the ***unit ball*** *centered at x if $r = 1$. A set $N(x)$ is called a* ***neighborhood*** *of $x \in X$ if there exists an $r > 0$ such that $B_r(x) \subseteq N(x)$.*

This seems fairly straight-forward. The open ball is just the set of all points in our space within the specified distance r.

Example: Consider the metric spaces $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_2)$ where

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

and

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Draw the unit balls centered at $\mathbf{0} = (0,0)$ in the (x_1, y_2)-plane.

Solution: Respectively, we are interested in the following sets:

$$d_1: \quad B_1(\mathbf{0}) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| < 1\right\}$$

and

$$d_2: \quad B_1(\mathbf{0}) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} < 1\right\}.$$

We recognize the second set as corresponding to the inside of a circle of radius 1 (excluding the boundary). The first set is a little more obscure, but we should be able to agree that it corresponds to the figure given in Figure 1(a). We should be quite satisfied the correspondence of the ball with metric d_2 and the traditional notion of a ball. For the metric d_1, the shape of the "ball" is a diamond! Nevertheless, we notice that the equations for the ball is analogous to our conventional understanding of the equations for a ball.

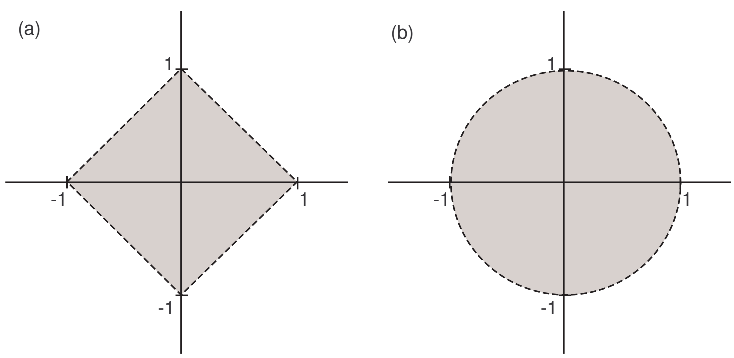

Figure 1: Unit balls in $\mathbb{R}^2$ for the metrics (a) d_1, and (b) d_2. The dotted edges indicate that the boundary points are not included in the ball.

Example: Consider the metric space $(\mathbb{R}^2, d_2)$. Show that the half-plane

$$H = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\right\}$$

is a neighborhood of $\mathbf{x} = (0, 1)$ but not of $\mathbf{y} = (0, 0)$.

Solution: All we need to do is find a ball centered at $\mathbf{x} = (0, 1)$, i.e. a $B_r(\mathbf{x})$, such that $B_r(\mathbf{x}) \subseteq H$. Since we clearly can take $r = 1/2$ and have $B_{1/2}(\mathbf{x}) \subseteq H$, it follows that H is a neighborhood of $\mathbf{x}$. It is clear that the half-plane H is not a neighborhood of $\mathbf{y} = (0,0)$ since every open ball $B_r(\mathbf{y})$ contains some $\mathbf{x} \in B_r(\mathbf{y})$ such that $x_2 < 0$. It follows that $\mathbf{x} \notin H$ so that H is not a neighborhood of $\mathbf{y}$. (See Figure 2.)

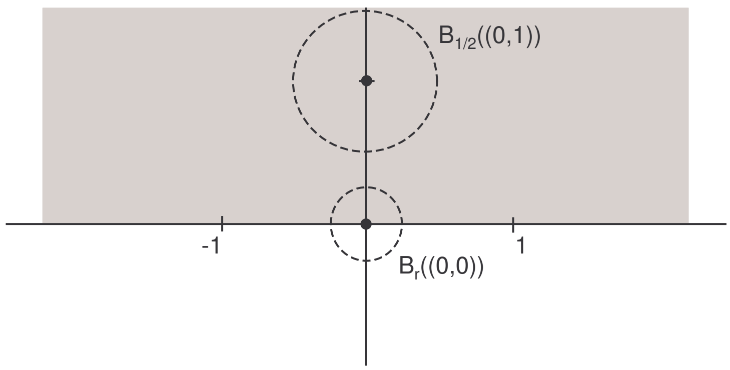

Figure 2: The upper half-plane H is a neighborhood of $\mathbf{x} = (0,1)$ but not of $\mathbf{y} = (0,0)$ since there is a ball around $\mathbf{x}$ entirely contained in H but no such ball around $\mathbf{y}$.

We introduce the following definitions.

Definition 1.2. *Let (X,d) be a metric space and suppose $S \subseteq X$. Then a point $p \in S$ is said to be an **interior point** of S if there is a neighborhood of p, $N(p)$, such that $N(p) \subseteq S$. A set S is said to be an **open set** if every point $p \in S$ is an interior point of S.*

Notation: It is common to denote the set of all interior points of a set S by S^o. Adopting this notion, the condition for a set being an open set is $S = S^o$.

Example: Consider the half-plane $H \subset \mathbb{R}^2$ and the points $\mathbf{x} = (0,1)$ and $\mathbf{y} = (0,0)$. We can now see that $\mathbf{x}$ is an interior point of H while $\mathbf{y} = (0,0)$ is not. Since not every point $\mathbf{z} \in H$ is an interior point it follows that H is *not open*.

Example: Show that the interval $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is an open set in the metric space $(\mathbb{R}, d_2)$.

Note: In $\mathbb{R}$, the three metrics d_1, d_2, and d_∞ coincide. Specifically, we have that $d_1(x, y) = d_2(x, y) = d_\infty(x, y) = |x - y|$.

Solution: The intuitive justification is clear. No matter where we are along the interval, $x \in (0, 1)$, we may construct an interval around x sufficiently small so that the new interval is contained in $(0, 1)$. Formally, we define

$$r_1 = \min \{x, 1 - x\} .$$

This just keeps track of whether we are close to the left or the right endpoint (0 or 1, respectively). Note that we have $r_1 \leq x$ so that $x - r_1 \geq 0$ and $r_1 \leq 1 - x$ so that $x + r_1 \leq 1$. It can then be easily seen that

$$B_{r_1}(x) = \{y \in \mathbb{R} \mid x - r_1 < y < x + r_1\} \subseteq (0, 1)$$

because $0 \leq x - r_1 < x < x + r_1 \leq 1$. □

We are not quite done yet. After all, we know that not all points are interior points, and that not all sets are open. We therefore define the following sets of points.

Definition 1.3. *Let (X, d) be a metric space and suppose $S \subseteq X$. Then a point $p \in X$ is said to be:*

1. *a **limit point** (or **accumulation point**) of S if every neighborhood $N(p)$ of p contains a point $q \in S$, $q \neq p$;*

2. *a **boundary point** of S if every neighbourhood $N(p)$ of p contains a point $q \in S$ and a point $q' \in S^c$; and*

3. *an **isolation point** of S if $p \in S$ and p is not a limit point.*

*A subset S is said to a **closed set** if it contains all of its limit points.*

Notation: It is common to denote the set of all limit points of S as S' and the set of all boundary points of S as ∂S. The condition for a set being a closed set is equivalent to $S' \subseteq S$.

It is important to notice that, when considering limit points, we must consider points *not in* S as well as those in S. In particular, there may be points in S^c which as "close enough" to S so that every neighborhood contains points in S.

It is also important to note that the neighborhoods in question *exclude* the point $p \in S$. That is to say, we consider true neighborhoods "around" a point. A consequence of this is that it is *not sufficient* for $p \in S$ to possess a limiting sequence of points which converges to p in order to conclude that p is a limit point! (Since this would imply that an isolated point $p \in S$ would be a limit point since the limiting sequence $\{p, p, p, \ldots\}$ converges to it. It is important that we exclude p from any such sequence!)

Example 1: Show that the upper-half plane H is a closed set.

Solution: We need to identify the limit points of H, H'. We notice immediately that, for all $\mathbf{x} \in H$, we have that $B_r(\mathbf{x})$ contains points in H other than $\mathbf{x}$. It follows that $H \subseteq H'$.

There may, however, be limit points *outside* of H. To explore this possibility, we consider the set

$$H^c = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 < 0 \right\}.$$

For any $(x_1, x_2) \in H^c$, take $r = x_2/2 > 0$ so that $x_2 < r < 0$. It is clear that $B_r(\mathbf{x}) \subseteq H^c$ so that $B_r(\mathbf{x})$ does not contain any points in H. It follows that $\mathbf{x} \notin H'$. Since the choice of $\mathbf{x} \in H^c$ was arbitrary, we have that $H' \subseteq H$ (actually, $H' = H$ in this case) so that H is a closed set. $\square$

Example 2: Show that the interval $I = (0, 1) \subseteq \mathbb{R}$ is not a closed interval.

Solution: We first consider $x \in I$. By an earlier argument, we can choose $r_1 = \min\{x, 1 - x\}$ so that $B_{r_1} = (x - r_1, x + r_1) \subseteq (0, 1)$. It follows that $I \subseteq I'$.

Now consider $x \in I^c$. It is clear that if $x < 0$ or $x > 1$, we can pick an $r > 0$ sufficiently small so that $B_r(x) \subseteq I^c$. If we choose $x = 0$ or $x = 1$, however, things are not quite so clear. In fact, for any $r > 0$, we have that there are $y, z \in I$ such that $0 < y < r'$ and $1 - r' < z < 1$ so that $B_{r'}(0) \cap I \neq \emptyset$ and $B_{r'}(1) \cap I \neq \emptyset$. It follows that $\{0, 1\} \in I'$ even though $\{0, 1\} \notin I$. It follows that I is *not* a closed set. $\square$

One thing that should have struck us with these two examples is that we considered properties of the set's complement. So what can we say about the complement of a set as it relates to interior points, limit points, open and closed sets? We have the following result.

Theorem 1.1 (Theorem 2.23 in Rudin)**.** *Suppose* (X, d) *is a metric space. Suppose* $S \subseteq X$*. Then:*

(1) If S *is an open set,* S^c *is a closed set.*

(2) If S *is a closed set,* S^c *is an open set.*

Proof. We will prove both cases by contradiction.

Proof of (1): Suppose S is open and S^c is not closed. Since S is open, we have that $x \in S$ implies there is an $r > 0$ so that $B_r(x) \subseteq S$. Now suppose that S^c is not closed. If S^c were closed, that would imply that $x \in (S^c)'$ implies $x \in S^c$. Since S^c is not closed, we have that there is an $x \in (S^c)'$ such that $x \in S$. It follows that there is an $x \in S$ so that $B_r(x)$ contains points in S^c for all $r > 0$. This is a contradiction with the fact that S is open. The claim (1) follows.

Proof of (2): Suppose S is closed and S^c is not open. Since S is closed, we have that S contains all of its limit points, so that if $x \in X$ has the property that $B_r(x)$ contains points in S for all $r > 0$, then $x \in S$. Now consider the claim that S^c is not open. If S^c were open, we would have $x \in S^c$ implies there is an $r > 0$ so that $B_r(x) \subseteq S^c$. It follows that if S^c is not open, then there exists an $x \in S^c$ such that $B_r(x)$ contains points in S for all $r > 0$. It follows that we have found an $x \in S^c$ such that $x \in S'$, contradicting the fact that S is closed. The claim (2) follows. □

Example: We can see that this was exactly how our previous examples worked out. We showed that the interval $I = (0, 1)$ was open, and now we can say that the set

$$I^c = (-\infty, 0] \cup [1, \infty) = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 1\}$$

is closed. We also saw that the upper half plane

$$H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$$

was closed, and we can clearly see that the complement set

$$H^c = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 < 0\}$$

is open.

Before we get too carried away with our newfound topological wisdom, we should make the following points:

- *Sets can be neither open nor closed!* Consider the interval

$$I = (0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\} .$$

  This set cannot be open, because $1 \in I$ but 1 is not an interior point, and this set cannot be closed, because 0 is a limit point but $0 \notin I$. It follows that the set is neither open *nor* closed.

- *Sets can be both open and closed!* Let's be careful about what this means. We have that $S \subseteq X$ is closed if $S' \subseteq S$, and that S is open if $S \subseteq S'$. To have *both* be true at the same time, we must have $S' \subseteq S \subseteq S^o$, i.e. every limit point is an interior point. For every metric space (X, d), we can immediate find one such set: $S = X$. For instance, if our space is $\mathbb{R}$, we can simply take $S = \mathbb{R}$. A consequence of Theorem 1.1 is that the set $S = \emptyset$ is also both open and closed. (We can also establish this directly by noting that the condition $x \in S'$ implies $x \in S$, and $x \in S$ implies $x \in S^o$ are vacuously true because all of the sets are empty.) We will consider the possibility of metric spaces for which sets other than $S = X$ and $S = \emptyset$ may be both open and closed when we consider *connected sets*.

2 Unions and Intersections

We understand what it takes for a set to be *open* or *closed* but we have yet to consider how such sets interact with one another. In particular, we will be interested in how the property of whether a set is open or closed is affected by the union and intersection operations.

Consider the example of the family of sets $\{S_\alpha\}$ where

$$S_\alpha = \{x \in \mathbb{R} \mid -\alpha < x < \alpha\} = (-\alpha, \alpha)$$

and $\alpha \in A$. We can easily determine, for any finite subfamily $A = \{\alpha_1, \ldots, \alpha_N\}$ drawn from the interval $(0, 1)$, we have that

$$\bigcup_{\alpha \in A} S_\alpha = S_m = (-M, M)$$

and

$$\bigcap_{\alpha \in A} S_\alpha = S_M = (-m, m)$$

where $m = \min(A)$ and $M = \max(A)$. In particular, we notice that the $\{S_\alpha\}$ is a family of *open* sets and that the union and intersection of the entire family are *open* sets. We might conjecture, therefore, that the finite intersection and union of open sets is an open set.

We now consider $A = (0, 1)$. As expected, we have that, we have that

$$\bigcup_{\alpha \in A} S_\alpha = (-1, 1)$$

which is an open set. We notice, however, that

$$\bigcap_{\alpha \in A} S_\alpha = \{0\}$$

is a *closed* set. So the infinite union produced an open set but that the infinite intersection produced a *closed* set. The property switched!

We have the following result.

Theorem 2.1 (Theorem 2.24(a,c) in Rudin)**.** *Suppose (X, d) is a metric space and $\{S_\alpha\}$ is a family of open sets such that $S_\alpha \subseteq X$ for all $\alpha \in A$. Then:*

(1) Every (finite or infinite) union from $\{S_\alpha\}$ is an open set.

(2) Every finite intersection from $\{S_\alpha\}$ is an open set.

Proof. We prove this directly.

Proof of (1): Define $S = \bigcup_{\alpha \in A} S_\alpha$. Suppose $x \in S$. It follows that $x \in S_{\alpha'}$ for some $\alpha' \in A$ and $S_{\alpha'}$ is an open set. It follows that there is an $r > 0$ such that $B_r(x) \subseteq S_{\alpha'}$. Clearly $B_r(x) \subseteq S$ so that x is an interior point of S. Since the choice of x was arbitrary, it follows that every point in S is an interior point, so that S is an open set.

Proof of (2): Define $S = \bigcap_{\alpha \in A} S_\alpha$. Suppose $x \in S$ where S_α is a finite family of open sets. It follows that, for every $\alpha \in A$, we have that there is an $r_\alpha > 0$ so that $B_{r_\alpha}(x) \subseteq S_\alpha$. Take

$$R = \min_{\alpha \in A} \{r_\alpha\}.$$

Since A is a finite set, we know that R exists. It follows that $B_R(x) \subseteq S$. Since x was chosen arbitrarily, it follows that every point $x \in S$ is an interior point of S, so that S is open. □

This is excellent! We can now close the book open sets, aside from the very special case of an infinite intersection. Furthermore, we can see in the proof exactly where the argument breaks down if we do consider the infinite case, since $\min_{\alpha \in A} r_\alpha$ may not exist if A is not finite (but if it does, the argument still holds!).

But what about *closed* sets? Consider the family of sets $\{S_\alpha\}$ where

$$S_\alpha = \{x \in \mathbb{R} \mid -\alpha \le x \le \alpha\}.$$

Proceeding as before, we have that any finite family $A = \{\alpha_1, \ldots, \alpha_N\}$ drawn from $(0, 1)$ gives

$$\bigcup_{\alpha \in A} S_\alpha = S_m = [-M, M]$$

and

$$\bigcap_{\alpha \in A} S_\alpha = S_M = [-m, m]$$

where $m = \min(A)$ and $M = \max(A)$. We can clearly see that both of these sets are closed.

Now consider $A = (0, 1)$. We have

$$\bigcap_{\alpha \in A} S_\alpha = S_1 = \{0\}$$

which is a closed set. For the union, we have to be careful, since we may approach the endpoints -1 and 1 but may never reach them (and therefore contain them in any set in the family). We have that

$$\bigcup_{\alpha \in A} S_\alpha = (-1, 1).$$

We notice immediately, however, that this is an *open* set. The property (again!) has switched.

We have the following result.

Theorem 2.2 (Theorem 2.24(b,d) in Rudin)**.** *Suppose (X, d) is a metric space and $\{S_\alpha\}$ is a family of closed sets such that $S_\alpha \subseteq X$ for all $\alpha \in A$. Then:*

(1) Every finite union from $\{S_\alpha\}$ is a closed set.

(2) Every (finite or infinite) intersection from $\{S_\alpha\}$ is a closed set.

Proof. We prove this directly.

Proof of (2): Define $S = \bigcap_{\alpha \in A} S_\alpha$ where $\{S_\alpha\}$ is a family of closed sets. We will show $x \in S'$ implies $x \in S$. Suppose $x \in S'$. It follows that, for every $r > 0$, $B_r(x)$ contains elements in S, and therefore contains elements in *each* S_α, $\alpha \in A$. It follows that $x \in S'_\alpha$ for all $\alpha \in A$. Since S_α is closed for all $\alpha \in A$, it then follows that $x \in S_\alpha$ for all $\alpha \in A$ so that $x \in S$. Since $x \in S'$ was chosen arbitrarily, the result follows.

Proof of (1): Define $S = \bigcup_{\alpha \in A} S_\alpha$ where $\{S_\alpha\}$ is a finite family of closed sets. Suppose $x \in S'$. By definition, for every $r > 0$, $B_r(x)$ contains elements in S. Since $\{S_\alpha\}$ is finite, there is a set $S_{\alpha*}$ such that $B_r(x)$ contains elements in $S_{\alpha*}$ for all $r > 0$. (To see this, suppose otherwise. Then there would be an $R > 0$ so that $B_r(x)$ does not contain elements in any S_α for all $0 < r < R$. This would imply that $B_r(x)$ does not contain any elements in S, a contradiction.) It follows that $x \in S'_{\alpha*}$ and, because $S_{\alpha*}$ is closed, it follows that $x \in S_{\alpha*}$. Consequently, we have that $x \in S$. Since the choice of x was arbitrary, the result follows. □

3 Closure and Interior

In various applications, we will be interested in only *open* or *closed* sets. That is to say, we will not be able to perform the operations we wish to perform without first ensuring we are dealing with either and open or closed set. We have already seen, however, that not every set is open or closed. So what can we do?

This issue is not as tricky as it seems. We will simply *construct* the sets we require. Furthermore, we already have one of the sets we need! If we are interested in making an open set out of S, we can throw away all of the points S which are not interior points. That is say, we pick S^o.

Theorem 3.1. *Suppose (X, d) is a metric space and $S \subseteq X$. Then the largest open set contained in S is S^o.*

Proof. We know that $S^o \subseteq S$ and S^o is open. It remains only to show that there is no larger set with these properties. Suppose otherwise. That is to say, suppose there is an $x \notin S^o$ so that the set $B = S^o \cup \{x\}$ satisfies $B \subseteq S$ and B is open. If $B \subseteq S$ and B is open, then $x \in S^o$, which is a contradiction. The result is shown. □

So, if we are interested in dealing with the open portion of S, we deal with the interior. Now suppose we are interested in constructing a closed set out of S. We have the following definition.

Definition 3.1. *Suppose (X, d) is a metric space and $S \subseteq X$. The **closure** of S, $\overline{S}$, is the set S together with its limit points, i.e. $\overline{S} = S \cup S'$.*

Theorem 3.2. *Suppose (X, d) is a metric space and $S \subseteq X$. Then the smallest closed set containing S is $\overline{S}$.*

Proof. It is clear that $S \subseteq \overline{S}$ and $\overline{S}$ is closed (homework). It remains only to show that there is no smaller set with these properties. Suppose otherwise. That is to say, suppose there is an $x \in \overline{S}$ such that $B = \overline{S} \setminus \{x\}$ satisifes $S \subseteq B$ and B is closed. Clearly, however, if $x \in S'$ then B is not closed, and if $x \in S \setminus S'$ then B does not contain S (since $x \in S$). The result follows. □

So if we need to make a closed set out of S, we need to consider the closure.

Example: Determine S^o and $\overline{S}$ for the following sets:

(1) $S = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

(2) $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$

Solution (1): We can fairly quickly determine that the interior points of S lie in the interval $(0, 1)$ and that the limit points are contained in $[0, 1]$. It follows that

$$\begin{aligned} S^o &= \{x \in \mathbb{R} \mid 0 < x < 1\} \\ \overline{S} &= \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}. \end{aligned}$$

Solution (2): We have not considered sets defined like this in a little while, but there is no reason we cannot. We have

$$S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}.$$

Consider the interior points. For every $x \in S$, we need to find an $r > 0$ so that $B_r(x) \subseteq S$. It is certainly clear that no such ball can be constructed around 1 or 1/2. How about further points, when n is large and the set begins to "bunch up"? We still have the same problem. No matter how far along the set we go, we will always have points in any ball which are not from S. It follows that there are no interior points, so that

$$S^o = \emptyset.$$

Now find the limit points. We need to find points $x \in \mathbb{R}$ so that $B_r(x)$ intersects S for all $r > 0$. We attempt first to find such points from S itself. We quickly grow frustrated. Carrying the earlier discussion forward, for any $x \in S$, we can find an $r > 0$ such that $B_r(x)$ is entirely disjoint from x! So, in fact, every point $x \in S$ is an *isolated* point, and therefore cannot be a limit point.

We are not done, however, since we need to consider $x \in S^c$ as well. We can quickly see that, for any $r > 0$, there is an $n \in \mathbb{N}$ so that $0 < 1/n < r$. It follows that $0 \in S'$ (even though $0 \notin S$). This is the only such point, so that we have

$$S' = \{0\}$$

so that

$$\overline{S} = S \cup S' = S \cup \{0\}.$$

4 Connected Sets

Consider the earlier question of whether a subset $S \subseteq X$ where (X, d) is a metric space can be both open *and* closed. We established that $S = X$ and $S = \emptyset$ were both open and closed, but left open the question whether there were any other such sets. Consider now the following example.

Example: Consider the metric space (X, d) where

$$X = \{x \in \mathbb{R} \mid 0 \leq x \leq 1 \text{ or } 2 \leq x \leq 3\}.$$

Show that the set $S = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is both open and closed.

Solution: This is straight-forward. We clearly have that, for every $x \in [0, 1]$, $B_{0.5}(x)$ only intersects points in S so that every $x \in [0, 1]$ is an interior point of S. (Notice we do not worry about intersecting with points $-0.5 < y < 0$ or $1 < y < 1.5$ becaus they are not contained in X.) Similarly, it is clear that every, for every $x \in [0, 1]$ and $r > 0$, $B_r(x)$ intersects $[0, 1]$ so that every point is a limit point, and there $[0, 1]$ is a closed set. It follows that S is both open and closed relative to the topology of (X, d). □

If this example seems manufactured to you, you are probably not alone. After all, in order to have a set be both open and closed, we needs to have $S' \subseteq S$ (closed) and $S = S^o$ (open) so that $S' \subseteq S^o$, i.e. every limit point is an interior point. That means that, if we can approach a point (limit point), we have to be entirely within the set (interior point). It seems like we should be able to get *everywhere* in the set by moving toward limits, and expanding out.

The only way to overcome this was to introduce "gaps" in the fabric of the metric space itself. We close off this possibility with the following definition.

Definition 4.1 (Definition 2.45 in Rudin)**.** *Suppose (X, d) is a metric space and $S_1, S_2 \subseteq X$. Then we will say that S_1 and S_2 are **separated** if $\overline{S}_1 \cap S_2 = \emptyset$ and $S_1 \cap \overline{S}_2 = \emptyset$. We will say a set $S \subseteq X$ is **connected** if there do not exist nonempty separated sets $S_1, S_2 \subseteq X$ such that $S = S_1 \cup S_2$.*

This is a little cumbersome, but we can certainly make sense out of it. For two sets S_1 and S_2 to be separated, we must have that $S_1 \cap S_2 = \emptyset$ and that no limit from within S_1 reaches S_2, and no limit from within S_2 reaches S_1. In order to be connected, therefore, it must be the case that every pair of subsets of S_1 and S_2 such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$ satisfies either $S_1' \cap S_2 \neq \emptyset$ or $S_1 \cap S_2' \neq \emptyset$.

Examples: We should be somewhat careful in applying this definition. Consider the determining whether the following subsets of $\mathbb{R}$ are connected.

(1) $S = \{x \in \mathbb{R} \mid 0 < x < 1 \text{ or } 1 < x < 2\}$

(2) $S = \{x \in \mathbb{R} \mid x = 0\}$

(3) $S = \mathbb{Q}$

Solution (1): Intuition should play a key role here. Since we can see that S has a "hole" in it, we suspect it should *not* be connected. But we have to check the definition. The obvious choices for the separated sets are $S_1 = (0, 1)$ and $S_2 = (1, 2)$. We can see that $\overline{S}_1 = [0, 1]$ and $\overline{S}_2 = [1, 2]$ and that, even though $\overline{S}_1 \cap \overline{S}_2 = \{1\}$, we have that $\overline{S}_1 \cap S_2 = \emptyset$ and $S_1 \cap \overline{S}_2 = \emptyset$. (Note, in particular, that S_1 and S_2 may still be separated even if $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$. That is to say, sharing limit points is not enough to overcome a "gap"!)

Solution (2): Intuition again tells us that this set S, although simple, is connected (albeit just the single element to itself). It is also clear that there are no separated sets such that $S_1 \cup S_2 = S$.

Solution (3): Intuition does not play as large a role here as before. After all, we know that between any two rational numbers, there are an infinite number of rational numbers (suggesting connectedness!), but also an infinite number of irrational numbers (suggesting separation!). We have to rely on the definition.

To construct our sets S_1 and S_2, we use the same trick we used earlier. Let's divide $\mathbb{Q}$ into two halves, with the value separating them *not* in the rational numbers. For instance, we can choose:

$$S_1 = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$$
$$S_2 = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}.$$

It is clear that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = \mathbb{Q}$ (since $\sqrt{2} \notin \mathbb{Q}$!) so this is at least a candidate decomposition of S. We notice that every $x \leq \sqrt{2}$ is a limit point since every ball $B_r(x)$ contains rational numbers. We therefore have $\overline{S}_1 = (-\infty, \sqrt{2}]$. This does not intersect S_2 so that $\overline{S}_1 \cap S_2 = \emptyset$. Similarly, we can see that $\overline{S}_2 = [\sqrt{2}, \infty)$ so that $S_1 \cap \overline{S}_2 = \emptyset$. It follows that S_1 and S_2 are separated sets and therefore that S is not connected.

Returning to our original question, we have the following result.

Theorem 4.1. *Suppose (X, d) is a metric space. Then the only subsets $S \subseteq X$ which are both open and closed are $S = X$ and $S = \emptyset$ if and only if X is connected.*

Proof. Homework! □