MATH 521, WEEK 10: Series, Partial Sums, Convergence

1 Series

Some of the most important applications of the results for sequences in the real numbers are with respect to analyzing *series*. That is to say, summations where the elements being summed are the terms in the sequence. Many examples spring to mind, from the arithmetic and geometric series used frequently in grade school mathematics, to power, Taylor, and Fourier series often utilized in advanced mathematics (and the applied sciences).

Our approach here will be brief, but (reasonably) rigorous. We start by formally defining the following.

Definition 1.1. *Consider a sequence $\{a_n\}$ of real numbers. We define the n^{th}* ***partial sum*** *of $\{a_n\}$ to be*

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n. \tag{1}$$

We define the ***infinite series*** *of $\{a_n\}$ to be*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots . \tag{2}$$

In earlier courses, we paid little attention to the subtleties of what (2) really means. After all, we know properties may change when extending from a finite case to an infinite one. In this case, we know that, for any finite number of real numbers, the sum (1) must give another real number. We can quickly convince ourselves that this is not true for the *infinite* series, since we can take

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots .$$

Whatever this is, it is not a real number. Suppose, however, that we insist a series remains *bounded*. Can we make sense of the infinite sum in this case? Well, consider the series

$$\sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + 1 + \cdots .$$

This does not tend to infinity but, whatever it is, it is definitely not a real number. So we do need to give a little thought to when the infinite series is a real number, i.e. when we have an $s \in \mathbb{R}$ such that

$$s = \sum_{n=1}^{\infty} a_n. \tag{3}$$

The resolution comes from considering the topic we just completed: convergence and divergence of sequences. The sequences we will consider, however, are not the original sequences $\{a_n\}$ but rather the sequences of partial sums $\{s_n\}$. Intuitively, as we sum more and more terms in a series a_n together, we should get closer to the infinite summation value s. Consider the following definition.

Definition 1.2. *Consider a sequence $\{a_n\}$ of real numbers and the resulting sequence of partial sums $\{s_n\}$ given by (1). We we will say (2) converges if there is an $s \in \mathbb{R}$ such that, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies that $|s_n - s| < \epsilon$.*

Otherwise, we will say that the series *diverges*.

Notice that if the sequence of partial sums converges, we are justified in writing (3). Recasting this problem as a convergence problem for the sequence of partial sums $\{s_n\}$ allows us to use the results already established for more conventional sequences. In particular, since we are working with real numbers, we may use properties such as monotonicity.

Furthermore, we recognize that convergence is equivalent to the Cauchy sequence criteria (i.e. the elements in the sequence become arbitrarily close together). For series, we will have to be careful when recognize what the sequence elements are which become close together. They are the *partial sums*. That is to say, for $m > n$, we need to consider bounding

$$|s_m - s_n| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right|.$$

Adjusting the indexes, the equivalent Cauchy convergence statement for series is the following.

Theorem 1.1 (Theorem 3.22 in Rudin)**.** *The infinite series corresponding to the sequence $\{a_n\}$ converges if and only if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m > n \geq N$ implies that*

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

Corollary 1.1. *If the infinite series of the sequence* $\{a_n\}$ *converges to some value* s*, then* $\{a_n\}$ *converges to zero.*

Proof. Take $m = n + 1$ in Theorem 1.1. It follows that, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_m| < \epsilon$ for $m > N$. That is to say, if the series converges to some value, then the terms in the underlying sequence must converge to zero. □

Note: The converse of Corollary 1.1 does not hold in general! That is to say, it is not enough that $a_n \to 0$ in order to conclude that $s_n \to s$ for some $s \in \mathbb{R}$.

Let's consider a few familiar examples.

Example 1: Show that the series with terms $a_n = \dfrac{1}{n}$, $n \in \mathbb{N}$, diverges.

Solution: There are a number of ways to prove this result. One of the most basic (because it requires no prior results) is the following. Consider dividing the sequence into successive groupings of size 2^n, $n \in \mathbb{N}$, elements. Specifically, consider the first and second term by themselves, the next two as a group, the next four as a group, and so on. We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\ &= (1) + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= (1) + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

Since we will never run out of groupings of length 2^n, no matter how far along the natural numbers we go, we must conclude that the sequence of partial sums is not bounded, and therefore (because it is monotone) the series does not converge. Note that this series does not converge even thought $\frac{1}{n} \to 0$ as $n \to \infty$. □

Example 2: Show that for $r \in \mathbb{R}$ such that $|r| < 1$ we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Solution: We have seen this result for the *geometric series* before, so we will go over the familiar details quickly. We will be careful now, however, to use the formal definition of series convergence.

We start by generating the sequence of partial sums $\{s_n\}$. We have

$$s_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n.$$

Notice that we also have

$$rs_n = r\sum_{k=0}^{n} r^k = r + r^2 + \cdots + r^n + r^{n+1}$$

so that

$$(1-r)s_n = 1 - r^{n+1} \quad \Longrightarrow \quad s_n = \frac{1-r^{n+1}}{1-r}.$$

We now want to prove that the sequence of partial sums converges to the limit $s = \dfrac{1}{1-r}$. Take $\epsilon > 0$. We have

$$|s_n - s| = \left| \frac{1-r^{n+1}}{1-r} - \frac{1}{1-r} \right| = \left| \frac{r^{n+1}}{1-r} \right|.$$

We want to bound this by ϵ. The intuition is fairly obvious: because $|r| < 1$, for any $\epsilon > 0$ we may choose n sufficiently large so that r^{n+1} gets vanishingly small. Formally, we invert the dependence on ϵ and n. We want

$$\left| \frac{r^{n+1}}{1-r} \right| < \epsilon \quad \Longleftrightarrow \quad |r|^{n+1} < \epsilon|1-r| \quad \Longleftrightarrow \quad n > \log_{|r|}\left(\epsilon|1-r|\right) - 1.$$

Note that the inequality in the logarithm has changed directions as a result of $|r| < 1$. It follows that, if we take $N > \log_{|r|}\left(\epsilon|1-r|\right) - 1$ and $n \geq N$, then we have

$$|s_n - s| = \left| \frac{r^{n+1}}{1-r} \right| \leq \left| \frac{r^{N+1}}{1-r} \right| < \epsilon$$

and we are done. $\square$

We will not give an exhaustive study of series in the real numbers here. We will, however, prove some of the basic convergence results which we have previously seen in various Calculus courses. In particular, we will consider the *comparison* and convergence/divergence of p-series. Time-permitting, we will investigate the well-known *ratio* and *root tests.*

2 Comparison Test

In general, it can be difficult to determine the convergence of series. The most basic tool for determining whether a series converges or not (and for proving more advanced results) is to compare to a more basic series which is known to either converge or diverge. This it not always guaranteed to work, but under special cases, we may be able to conclude either convergence or divergence for a given series.

We have the following result.

Theorem 2.1 (Comparison Test)**.** *Consider sequences* $\{a_n\}$ *an* $\{b_n\}$ *and corresponding series (a)* $\sum_{n=1}^{\infty} a_n$ *and (b)* $\sum_{n=1}^{\infty} b_n$*. Suppose there is an* $N' \in \mathbb{N}$ *such that:*

1. $|a_n| \leq b_n$ *for all* $n \geq N'$*. Then, if (b) converges, so does (a).*
2. $a_n \geq b_n \geq 0$ *for all* $n \geq N'$*. Then if (b) diverges, so does (a).*

Proof of 1.: We will use the Cauchy condition for convergence. Since $\{b_n\}$ converges, we have that, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m > n \geq N$ implies that $\sum_{k=n+1}^{m} b_k < \epsilon$. It follows that

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} b_k < \epsilon$$

where the first inequality follows by the triangle inequality on the real numbers, and the second follows from our assumption. It follows from Theorem 1.1 that $\sum_{n=1}^{\infty} a_n$ converges. □

Proof of 2.: Suppose that $\sum_{n=1}^{\infty} a_n$ converges and conditions of (b) hold. It would then follow from part (a) that $\sum_{n=1}^{\infty} b_n$ converges, which is a contradiction. The result follows. □

3 p-Series and Applications

Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

where $p > 0$. We have already seen that this series diverges for $p = 1$. How about other values of p? As motivation, let's consider the case of $p = 2$. We are considering the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

We will play a little trick (naturally, it is one that is easy to follow, but hard to find!). We might notice that we have

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

where the final step follows from partial fraction decomposition (although it is easier to just confirm it by working backwards).

We will now apply the Cauchy criterion for convergence. That is to say, we consider the summation from $k = n + 1$ to m where $m > n$ and wish to show the result can be bounded. We have

$$\begin{aligned}\sum_{k=n+1}^{m} \frac{1}{k^2} &< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} + \frac{1}{m}\right) \\ &= \frac{1}{n} - \frac{1}{m}\end{aligned}$$

where all the middle terms have cancelled. (A partial sum where the middle terms cancel in this fashion is called a *telescoping series*.) We now notice that, for any $\epsilon > 0$, if we pick $N > \frac{1}{\epsilon}$ we have that, for any $m > n \geq N$,

$$\left| \sum_{k=n+1}^{m} \frac{1}{k^2} \right| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

That is to say, the series satisfies the Cauchy criterion. It follows that the sequence converges.

So we have that the p-series diverges for $p = 1$ and converges for $p = 2$. It is a small step further to prove that it diverges for $p \leq 1$ and converges for $p \geq 2$ (comparison test!). How about the rest of the values?

We introduce the following preliminary result, known as the Cauchy Condensation Test.

Theorem 3.1 (Theorem 3.27 in Rudin)**.** *Suppose $\{a_n\}$ is a sequence of real numbers such that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \tag{4}$$

converges.

Aside from the showing that the convergence of many series depends on a particular sparse subset of the original series, it is often easier to show convergence for the sequence (4) than the original one.

Proof. Consider the partial sums

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_m = \sum_{k=0}^{m} 2^m a_{2^m}.$$

Notice that, for $n < 2^m$, we have that

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^m} + \cdots + a_{2^{m+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^m a_{2^m} \leq t_m. \end{aligned}$$

It follows that if s_n diverges, then t_m diverges, and also that if t_m converges, s_n converges (by monotonicity).

On the other hand, for $n > 2^m$, we have

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots (a_{2^{m-1}+1} + \cdots + a_{2^m}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{m-1} a_{2^m} = \frac{1}{2} t_m. \end{aligned}$$

It follows that if s_n converges, then t_n converges, and also that if t_m diverges, s_n diverges (again, by monotonicity). We are done. □

We are now (finally!) prepared to prove exactly when the p-series converges and diverges.

Theorem 3.2. *The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *diverges if* $0 < p \leq 1$ *and converges if* $p > 1$.

Proof. Since for all $p > 0$ we have

$$1 \geq \frac{1}{2^p} \geq \frac{1}{3^p} \geq \cdots \geq 0 \implies a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$$

we may apply the Cauchy Condensation Test. We have have the p-series converges if and only if

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converges, so that we check

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \left(2^{1-p}\right)^k.$$

This is a geometric series with parameter $r = 2^{1-p}$. We know this converges if $|r| < 1$, which corresponds to $p > 1$, and diverges for $|r| \geq 1$, which corresponds to $p \geq 1$. The proof is complete. □

Example: Prove that the series with terms

$$a_n = \frac{n}{n^2 - 1}$$

for $n \geq 2$ diverges.

Solution: We notice that the terms of the sequence $\{a_n\}$ become more and more like $\{\frac{1}{n}\}$ as n grows. Since the p-series with $p = 1$ diverges, we suspect some form of comparison may be in order. We check that

$$\frac{n}{n^2 - 1} - \frac{1}{n} = \frac{n^2 - n^2 + 1}{n(n^2 - 1)} = \frac{1}{n(n^2 - 1)} > 0$$

for all $n \geq 2$. It follows that

$$\frac{n}{n^2 - 1} > \frac{1}{n}.$$

By the comparison test, we conclude that

$$\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}$$

diverges.

4 Ratio and Root Tests (Time-Permitting)

In previous calculus courses, we have analyzed the convergence and divergence of series by appealing to the ratio and root tests. We have probably, however, done these with an emphasis on application rather than justification. We now fill in this gap.

Theorem 4.1 (Ratio Test)**.** *Consider a sequence $\{a_n\}$ with a corresponding series $\sum_{n=1}^{\infty} a_n$. Define $R = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and $r = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then:*

(a) If $R < 1$ then the series converges.

(b) If $r > 1$ then the series diverges.

The key distinction with previous definitions is that we now consider the more general lim inf and lim sup instead of the standard lim. This allows us to handle sequences which do not necessarily have a limit.

Proof. Suppose (a) holds. That is, suppose there is an α such that $R \leq \alpha < 1$ and

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\sup_{m\geq n} \left|\frac{a_{m+1}}{a_m}\right|\right) \leq \alpha.$$

It follows that there is an $N \in \mathbb{N}$ so that, for $n \geq N$, we have

$$\sup_{m\geq n} \left|\frac{a_{m+1}}{a_m}\right| \leq \alpha \quad \Longrightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha$$

for all $n \geq N$. It follows that

$$\begin{aligned} & |a_n| \leq \alpha |a_{n-1}| \leq \alpha^2 |a_{n-2}| \leq \alpha^3 |a_{n-3}| \\ \Longrightarrow \quad & |a_n| \leq \alpha^{n-N} |a_N|. \end{aligned}$$

Now consider the series with terms $b_n = \alpha^{n-N}|a_N|$. We have

$$\sum_{n=N}^{\infty} b_n = \sum_{n=N}^{\infty} \alpha^{n-N}|a_N| = \frac{|a_N|}{\alpha^N} \sum_{n=N}^{\infty} \alpha^n = \frac{|a_N|}{\alpha^N} \frac{\alpha^N}{1-\alpha} = \frac{|a_N|}{1-\alpha}$$

since the series of geometric with parameter $\beta < 1$. It follows by the comparison theorem that the series with terms a_n converges.

Now suppose (b) holds. It follows that there is an $r \geq \alpha > 1$ such that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\inf_{m\geq n} \left|\frac{a_{m+1}}{a_m}\right|\right) \geq \alpha.$$

It follows that there is an $N \in \mathbb{N}$ so that, for $n \geq N$, we have

$$\inf_{m \geq n} \left| \frac{a_{m+1}}{a_m} \right| \geq \alpha \quad \Longrightarrow \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \alpha$$

for all $n \geq N$. It follows that

$$\begin{aligned} & |a_n| \geq \alpha |a_{n-1}| \geq \alpha^2 |a_{n-2}| \geq \alpha^3 |a_{n-3}| \\ \Longrightarrow \quad & |a_n| \geq \alpha^{n-N} |a_N| > 0. \end{aligned}$$

Since it is clearly impossible that $\lim_{n \to \infty} a_n = 0$ in this case, and that this is necessarily for convergence, we have that the series diverges. ☐

Theorem 4.2 (Root Test)**.** *Consider a sequence $\{a_n\}$ with a corresponding series $\sum_{n=1}^{\infty} a_n$. Define $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Then:*

(a) If $\alpha < 1$ then the series converges.

(b) If $\alpha > 1$ then the series diverges.

Proof. Suppose (a) is satisfied. Following the method of Theorem 4.1, we have that there is an α satisfying $0 \leq \alpha < 1$ and an $N \in \mathbb{N}$ so that

$$\sqrt[n]{|a_n|} < \alpha$$

for all $n \geq N$. It follows immediately that

$$|a_n| < \alpha^n$$

and, since we know that

$$\sum_{n=N}^{\infty} \alpha^n = \frac{\alpha^N}{1 - \alpha}$$

since it is a geometric series with parameter $|\alpha| < 1$. It follows by the comparison theorem that the series with terms a_n converges.

Now suppose (b) holds. It follows that there is an $\alpha > 1$ and a subsequence $\{n_k\}$, $k \in \mathbb{N}$, such that

$$\lim_{k \to \infty} \sqrt[n_k]{|a_{n_k}|} = \alpha.$$

Notably, we can pick $\epsilon > 0$ sufficiently small so that

$$1 \leq \alpha - \epsilon < \sqrt[n_k]{|a_{n_k}|} < \alpha + \epsilon \quad \Longrightarrow \quad 1 \leq (\alpha - \epsilon)^{n_k} < |a_{n_k}|.$$

It follows that there are an infinite number of terms in $a_n > 1$ so that $\lim_{n \to \infty} a_n = 0$ is not possible. Since this is a necessarily condition for the convergence of the corresponding series, we have that the series diverges. ☐