

# MATH 521, WEEK 13:

## Derivatives, Mean Value Theorem, L'Hôpital's Rule

Many applications require an understanding of the *rate* at which one variable changes with respect to changes in another. For instance, we might be interested in changes in an object's position with respect to changes in time (i.e. its velocity), a bended beam's height relative to its length (i.e. its curvature), or when a stock portfolio may not improve its expected return with respect to changes in its investments (i.e. its optimal value).

Our task now is to understand this from a mathematical point-of-view. Our discussion will be brief, as most of these topics have been covered in some detail in previous calculus courses, and especially in Math 421. Nevertheless, we will prove a few results which we may have previously taken for granted, and emphasize a few applications of differentiation which may not have been previously obvious.

### 1 The Derivative

We have the following classical definition.

**Definition 1.1.** *Let  $f : I \mapsto \mathbb{R}$  where  $I \subseteq \mathbb{R}$  is an interval. We define the derivative of  $f$  at the value  $a \in I$  to be*

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}. \quad (1)$$

It is also common (although Rudin does not favor it) to let  $h = x - a$  and use the equivalent form

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

The form (1), however, allows us to make the immediate connection with limits as discussed previously. We have that (1) is satisfied if and only if, for every  $\epsilon > 0$ , there is a  $\delta > 0$  so that  $|x - a| < \delta$  implies that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon.$$

If the limit (1) exists at a given  $a \in \mathbb{R}$ , we will say that  $f$  is *differentiable* at  $a$ . Note that it is also common to use the notations

$$f'(a) = \frac{df}{dx}(a) = \left[ \frac{d}{dx} f(x) \right]_{x=a} = \dots$$

**Example 1:** Show that for  $f(x) = x^2$ , we have  $f'(x) = 2x$  for all  $x \in \mathbb{R}$ .

**Solution:** We will apply the definition, but we will be careful to use the rigorous definition of the limit. Take  $\epsilon > 0$  and suppose  $|x - a| < \delta$  (where  $\delta = \epsilon$ ). Then we have that

$$\begin{aligned} \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| &= \left| \frac{x^2 - a^2}{x - a} - 2a \right| \\ &= \left| \frac{(x - a)(x + a)}{x - a} - 2a \right| \\ &= |(x + a) - 2a| \\ &= |x - a| < \delta = \epsilon. \end{aligned}$$

Making the final substitution of  $a$  into  $x$  (a slight abuse of notation, but one which is widespread in practice), we have that  $f'(x) = 2x$ .  $\square$

We should be careful to recall that we are talking about a limit in the rigorous sense we have previously defined. We can take short-cuts with the example above, but must in general be more careful. Consider the following.

**Example 2:** Show that  $f(x) = |x|$  is not differentiable at  $x = 0$ .

**Solution:** Recall that

$$|x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0. \end{cases}$$

It follows that, for  $x > 0$  we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \frac{x}{x} = 1$$

while for  $x < 0$  we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \frac{-x}{x} = -1.$$

It follows that, for any  $\delta > 0$ , the interval  $(-\delta, \delta)$  contains points where the derivative quotient is 1 and points where it is  $-1$ . It is not true, therefore, that the quotient becomes arbitrarily close to any single value. Consequently, we have that

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist. □

## 2 Continuity and Differentiation

An immediate consequence of Example 2 above is that a continuous function need not be differentiable at every point in its domain. We might wonder what happens in the other direction. Is it possible for a function to be discontinuous despite being differentiable?

**Theorem 2.1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If  $f$  is differentiable at  $a \in \mathbb{R}$  then it is continuous at  $a$ .*

*Proof.* The rigorous  $\epsilon - \delta$  proof of this result is a little challenging, although it is similar to the argument we made last week to prove  $f(x) = x^3$  was continuous by decomposing the function into  $f(x) = x \cdot x^2$  and showing  $x$  and  $x^2$  are continuous, so their product must be. In this case, we will take for granted that the product of two limits is the product of the limits (Theorem 3.3 in Rudin).

In order to prove that  $f(x)$  is continuous at  $a \in \mathbb{R}$ , it is sufficient to show that

$$\lim_{x \rightarrow a} f(x) = f(a) \iff \lim_{x \rightarrow a} f(x) - f(a) = 0.$$

In this case, we have that

$$\begin{aligned} \lim_{x \rightarrow a} f(x) - f(a) &= \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} \right) (x - a) \\ &= \left[ \left( \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left( \lim_{x \rightarrow a} (x - a) \right) \right] \\ &= f'(a) \cdot 0 \\ &= 0 \end{aligned}$$

where we have used the fact that, if the limits exist, then the limit of a product is the product of the limits, and we are done. □

### 3 General Derivative Rules

We are familiar with the standard differentiation rules for real-valued functions from Calculus (and in more detail, if you took Math 421). We stop only briefly to re-iterate them here.

**Theorem 3.1.** *Suppose  $f : I \mapsto \mathbb{R}$  and  $g : I \mapsto \mathbb{R}$  are differentiable on some interval  $I \subseteq \mathbb{R}$ . Then, for every  $x \in I$  we have:*

1.  $\frac{d}{dx} [cf(x)] = cf'(x)$  for all  $c \in \mathbb{R}$ ;
2.  $\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x)$ ;
3.  $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$ ; and
4.  $\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ , provided  $g(x) \neq 0$ .

*Proof.* The proofs of (a) and (b) are trivial.

*Proof of (c):* We have that

$$\begin{aligned} \frac{d}{dx} [f(x)g(x)] &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x} \\ &= \left[ \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \right] \lim_{t \rightarrow x} g(t) + f(x) \left[ \lim_{t \rightarrow x} \frac{g(t) - g(x)}{t - x} \right] \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

where we have use the definitions of  $f'(x)$  and  $g'(x)$  and the fact that the limit of products is the product of the limits.

*Proof of (d):* We have that

$$\begin{aligned}
 \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= \lim_{t \rightarrow x} \frac{1}{t-x} \left[ \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} \right] \\
 &= \lim_{t \rightarrow x} \frac{1}{t-x} \left[ \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)} \right] \\
 &= \lim_{t \rightarrow x} \frac{1}{t-x} \left[ \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{g(t)g(x)} \right] \\
 &= \frac{1}{g(x)} \lim_{t \rightarrow x} \frac{1}{g(t)} \left[ \left( \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t-x} \right) g(x) - f(x) \lim_{t \rightarrow x} \frac{g(t) - g(x)}{t-x} \right] \\
 &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
 \end{aligned}$$

where, again, we have used the fact the limit of products if the product of the limits.  $\square$

Of course, this is not a complete list of all the derivative rules we have used in the various Calculus courses we have taken. Perhaps the most important is the following, which allows us to combine these rules.

**Theorem 3.2** (Chain Rule). *Suppose  $f : I_1 \mapsto I_2$  and  $g : I_2 \mapsto \mathbb{R}$  where  $I_1, I_2 \subseteq \mathbb{R}$  are intervals. Suppose  $f$  is differentiable at  $a \in I_1$  and  $g$  is differentiable at  $f(a) \in I_2$ . Then  $g(f(x))$  is differentiable at  $a$  and*

$$\frac{d}{dx} g(f(x)) = g'(f(a))g'(a).$$

**Note on common incorrect proof method:** It is tempting (but technically incorrect!) to write

$$\begin{aligned}
 &\left| \frac{g(f(x)) - g(f(a))}{x-a} - g'(f(a))f'(a) \right| \\
 &= \left| \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x-a} - g'(f(a))f'(a) \right| < \epsilon
 \end{aligned}$$

where we are able to provide the final bound by noting that

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} &= g'(f(a)) \\
 \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} &= f'(a)
 \end{aligned}$$

so that the limit converges because the product of limits is the limit of products.

This is actually not always correct! The reason is subtle, but worth noting. The problem comes from the observation that  $f(x) - f(a) = 0$  is possible for some functions regardless of the bound  $|x - a| < \delta$ . That is to say, we would actually be dividing by zero infinitely often as we take  $|x - a| \rightarrow 0$ , so that this portion of the limit does not converge to  $g'(f(a))$  as anticipated. This happens, for instance, for the function  $f(x) = x^2 \sin(1/x)$ , which is differentiable at  $x = 0$  provided we define  $f(0) = 0$ . In general practice, we ignore this technicality, but in analysis we cannot!  $\square$

*Proof.* Note that the derivative for  $g$  is with respect to changes in its argument, which is  $f(x)$ , not  $x$ .

First take  $\epsilon > 0$  and consider the derivative definition for  $f(x)$  at  $a \in I_1$ . We have

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$$

provided  $|x - a| < \delta$  for some  $\delta > 0$ . Rather than performing the analysis with  $\epsilon$  and  $\delta$  bounds, however, we make the following observation: there is a function which depends on  $x \in I_1$  and  $\epsilon$  which makes this into an equality. That is to say, there is an  $h(x, \epsilon)$  such that

$$\frac{f(x) - f(a)}{x - a} - f'(a) = h(x, \epsilon).$$

We further note that, by definition, we have  $h(x, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  (so the  $x$  dependence is not important). It follows that we have

$$f(x) - f(a) = (x - a)(f'(a) + h(x, \epsilon)).$$

Applying the same technique to  $g(y)$ ,  $y \in I_2$ , we have that

$$g(y) - g(b) = (y - b)(g'(b) + h(y, \epsilon)).$$

Again, we have that  $h(y, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  so that the  $y$  dependence is not important.

We now construct the proper differential quotient by taking  $y = f(x)$  and  $b = f(a)$ . We have

$$g(f(x)) - g(f(a)) = (x - a) [g'(f(a)) + h(f(x), \epsilon)] (f'(a) + h(x, \epsilon))$$

$$\implies \frac{g(f(x)) - g(f(a))}{x - a} = [g'(f(a)) + h(f(x), \epsilon)] (f'(a) + h(x, \epsilon))$$

Taking  $|x - a| \rightarrow 0$  so that  $\epsilon \rightarrow 0$ , we have that

$$\frac{d}{dx}g(f(x)) = \lim_{x \rightarrow a} \frac{g(f(x)) - g(f(a))}{x - a} = g'(f(a))f'(a).$$

□

**Examples:** Differentiate  $f(x) = \tan^2(x)$ .

**Solution:** Assuming we are allowed to use the rules for basic functions from Calculus (e.g.  $\frac{d}{dx} \sin(x) = \cos(x)$ ), we can simply apply the rules. In this case, we will choose to write  $\tan(x) = \frac{\sin(x)}{\cos(x)}$ . We have

$$\begin{aligned} \frac{d}{dx} \left[ \frac{\sin(x)}{\cos(x)} \right]^2 &= 2 \left[ \frac{\sin(x)}{\cos(x)} \right] \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right) \\ &= 2 \left[ \frac{\sin(x)}{\cos(x)} \right] \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= 2 \frac{\sin(x)}{\cos^3(x)} \\ &= 2 \tan(x) \sec^2(x). \end{aligned}$$

(Of course, we could have used the simplified rule  $\frac{d}{dx} \tan(x) = \sec^2(x)$  directly, as well.)

## 4 Mean Value Theorems

We should be familiar with the following result from earlier Calculus courses.

**Corollary 4.1** (Mean Value Theorem). *Suppose  $f : [a, b] \mapsto \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists a  $c \in (a, b)$  such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad (2)$$

That is to say, for every differentiable function on an interval, the average rate of change over the interval is realized as the instantaneous rate of change for at least one point in the interval (see Figure 1).

We can intuitively justify the mean value theorem by the following thought exercise. Imagine you are a state highway patrol officer and are

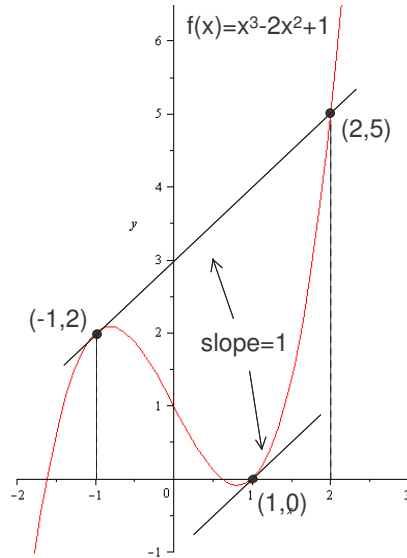


Figure 1: The instantaneous rate of change at  $x = 1$  is the same as the averaged rate of change over the interval  $(-1, 3)$  for  $f(x) = x^3 - 2x^2 + 1$ .

suspicious of a particular vehicle's erratic behavior. You clock it with your radar gun and are surprised, however, to find the questionable vehicle is traveling at 55 mph, just below the posted 60 mph limit. Nevertheless, you phone ahead to a fellow officer 80 miles down the interstate and give them a heads-up regarding your suspicions. Sure enough, in exactly an hour's time, the other officer spots the vehicle, but clocks it at 50 mph—again, below the posted 60 mph limit.

This should be the end of the story (and would likely be in practice), but you have taken a few mathematics courses and realize that, even though you did not *observe* the vehicle exceeding 60 mph, in order to travel 80 miles in an hour's time, it must necessarily have exceeded 60 mph *somewhere* in the stretch of highway between the two observations. In fact, to make up for the slower speed at the end points, you conclude that it safely *exceeded* 80 mph at some point during its trip. This is, in essence, exactly what the mean value theorem states! It is a relationship between the average rate of change over an interval and an instantaneous rate of change within the same interval. So even though we do not see the entire interval, we may conclude



something very insightful about the function's behavior there.

The mean value theorem is not restricted to applications such as this. In fact, even on the real number line, there are generalizations which prove very useful in a variety of applications! (We will derive a generalized form here which will be useful in proving the well known L'Hôpital's Rules.) First, we will need the following result.

**Theorem 4.1** (Rolle's Theorem). *Suppose  $f : [a, b] \mapsto \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then, if  $f(a) = f(b)$ , there is a point  $c \in (a, b)$  such that  $f'(c) = 0$ .*

*Proof.* We will use a little basic topology to simplify our work. We know, first of all, that since  $[a, b]$  is a compact set in  $\mathbb{R}$ , and that  $f$  maps into  $\mathbb{R}$ , that the maximum and minimum of  $f$  on  $[a, b]$  exist in the interval.

Suppose first of all that the maximum and minimum occur *only* at the endpoints  $a$  and  $b$ . It follows from  $f(a) = f(b)$  that we would have

$$\max_{x \in [a, b]} f(x) = \min_{x \in [a, b]} f(x) = f(a) = f(b)$$

so that  $f(x) = f(a) = f(b)$  for all  $x \in [a, b]$ . We can trivially see, therefore, that

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0$$

for all  $c \in [a, b]$ .

Now suppose at least one of the minimum and maximum occurs in  $(a, b)$ . Without loss of generality, suppose it is the maximum and occurs at  $c \in (a, b)$ . It follows that we have

$$f(c) \geq f(x) \implies f(x) - f(c) \leq 0$$

for all  $x \in [a, b]$ . It follows that, for every  $a \leq x < c$  we have

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

and for every  $c < x \leq b$  we have

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

Taking the limit as  $x \rightarrow c$  from both sides, we must therefore have

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0$$

and we are done. □

This result seems pretty straight-forward, but actually allows us to prove some very strong results. The first result may seem strange, but is actually incredibly useful. We will use it to prove both the Mean Value Theorem as we traditionally understand it, and also L'Hôpital's Rule.

**Theorem 4.2** (Generalized Mean Value Theorem). *Suppose  $f : [a, b] \mapsto \mathbb{R}$  and  $g : [a, b] \mapsto \mathbb{R}$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists a  $c \in (a, b)$  such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

*Proof.* The proof is actually incredibly simple, but slightly difficult to get started. We just need to apply Rolle's Theorem, but to what exactly?

The answer is to carefully construct a function which serves our needs. In this case, we select

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

We note a few things. First of all, the continuity of  $f$  and  $g$  on  $[a, b]$  guarantees the same for  $h$ . We also obtain differentiability on  $(a, b)$  in the same manner. More to the point, we notice that

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

That is to say,  $h : [a, b] \mapsto \mathbb{R}$  satisfies the assumptions of Rolle's Theorem. It follows that there is a  $c \in (a, b)$  such that

$$\begin{aligned} h'(c) &= [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0 \\ \implies \frac{f'(c)}{g'(c)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

and the result is shown. The standard Mean Value Theorem (Theorem 4.1) followed directly by taking  $g(x) = x$ .  $\square$

**Note on applications:** While we commonly interpret the Mean Value Theorem as providing information about  $f'(x)$ , we can also use it to get information about the original function  $f(x)$ . If we take the absolute value of (2) we arrive at

$$|f'(c)| = \left| \frac{f(b) - f(a)}{b - a} \right| \implies |f(b) - f(a)| = |f'(c)||b - a|. \quad (3)$$

What is intriguing about this form is that it explicitly relates two forms we have often been trying to relate in our studies of continuity, that is,  $|f(x) - f(a)|$  and  $|x - a|$ . For instance, consider the following example.

**Example:** Prove that  $f(x) = e^{-x}$  is uniformly continuous on  $[0, \infty)$ .

**Solution:** Note that  $f(x) = e^{-x}$  is continuous on  $[0, \infty)$  but that  $[0, \infty)$  is not compact, so we cannot use the result from last week. Instead, consider the following argument. Since  $f(x) = e^{-x}$  is differentiable on  $[0, \infty)$  and, specifically,  $f'(x) = -e^{-x}$ , we have by (3) that

$$|f(x) - f(y)| = |e^{-x} - e^{-y}| = |-e^{-c}||x - y| = e^{-c}|x - y|$$

for some  $c \in [0, \infty)$ . We know, however, that

$$\max_{c \in [0, \infty)} e^{-c} = 1$$

because it is monotonically decreasing on  $[0, \infty)$ . So we have  $e^{-c} \leq 1$  for all  $[0, \infty)$ . It follows, for every  $\epsilon > 0$ , if we take  $\delta = \epsilon$  then, for every  $x, y \in [0, \infty)$ ,  $|x - y| < \delta$  implies that

$$|f(x) - f(y)| \leq \max_{c \in [0, \infty)} \{e^{-c}\}|x - y| = |x - y| < \delta = \epsilon.$$

It follows that  $f(x) = e^{-x}$  is uniformly continuous on  $[0, \infty)$ . □

The moral of the story is that, if we can bound  $|f'(c)|$  on the interval  $(a, b)$ , we can get an explicit relationship on exactly *how* continuous the function is. This is very important in advanced analysis when we need stronger notions of continuity than simple continuity at a point (e.g. uniform continuity, Lipschitz continuity [not covered in Math 521], contraction mappings, etc.).

## 5 L'Hôpital's Rule

A primary application of the generalized mean value theorem is the following.

**Theorem 5.1** (L'Hôpital's Rule). *Suppose  $f : I \mapsto \mathbb{R}$  and  $g : I \mapsto \mathbb{R}$  are differentiable on an interval  $I \subseteq \mathbb{R}$  and  $g(x) \neq 0$  for any  $x \in I$ . Suppose there is an  $a \in \mathbb{R} \cup \{\pm\infty\}$  such that  $a \in I$  and*

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0. \tag{4}$$

Suppose furthermore that there is an  $L \in \mathbb{R} \cup \{\pm\infty\}$  such that

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L. \quad (5)$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

**Note:** The set  $\mathbb{R} \cup \{\pm\infty\}$  is sometimes called the *extended real number system*. The only difference with the real numbers as we have used them so far in this course is that we treat  $-\infty$  and  $\infty$  as *elements* in our system. That is, we can say things like  $a = \infty$ , and  $b = a$  implies  $b = \infty$ , without worrying about whether  $a$  and  $b$  are real numbers.

**Note:** We have not formally defined limits at  $\infty$  (or  $-\infty$ ) in this course. We should nevertheless be comfortably easily extending these notions. We have that

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every  $\epsilon > 0$ , there is an  $M > 0$  such that  $x > M$  implies  $|f(x) - L| < \epsilon$ . We also have that

$$\lim_{x \rightarrow a} f(x) = \infty$$

if, for every  $M > 0$ , there is a  $\delta > 0$  such that  $0 < |x - a| < \delta$  implies  $x > M$ . We can quickly modify these to apply to one-sided limits, limits at  $-\infty$ , etc.

*Proof.* Note first of all that the proof is much simpler we assume that  $f$  and  $g$  are *continuously differentiable*; that is, if the *derivatives* are continuous on  $I$ . In fact, you may have already seen the proof in this case in a previous Calculus course—it is approximately one line long. We will not, however need to assume this. So, unfortunately, we will have to do more work!

We start by making the simplifying assumption that  $a$  and  $L$  are finite real numbers. We will see that the argument easily extends to take limits where either (or both) of  $a$  and  $L$  are allowed to be infinite. For now, consider the right-sided limit

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L.$$

Take  $a < x$  and introduce an value  $y \in I$  such that  $a < y < x$ . Note that by (4) we have

$$\lim_{y \rightarrow a^+} \frac{f(x) - f(y)}{f(x) - g(y)} = \frac{f(x)}{g(x)}.$$

(At this point, we will allow that limits to be incorporated explicitly into expressions like this, rather than worrying significantly about exact  $\epsilon$  and  $\delta$  bounds!)

Now consider the form

$$\frac{f(x) - f(y)}{g(x) - g(y)}$$

in a little more depth. We know that  $f$  and  $g$  are differentiable on  $I$ , so that, from the generalized mean value theorem, there is a  $c \in (x, y)$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(y)}{g(x) - g(y)}.$$

Note now that we have  $a < y < c < x$  and that  $c \rightarrow a^+$  and  $y \rightarrow a^+$  as  $x \rightarrow a^+$  (so long as we can maintain this selection, which we always can!). Taking these limits simultaneously, we have that

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f(x) - f(y)}{g(x) - g(y)} = \lim_{x \rightarrow a^+} \frac{f'(c)}{g'(c)} = L$$

where we have used (5) in the final step, and we are done!

If we are worried about the formal  $\epsilon$ 's and  $\delta$ 's, we notice that, for any fixed  $a < y < c < x$  selected as above, we have

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - L \right| &= \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)} + \frac{f(x) - f(y)}{g(x) - g(y)} - \frac{f'(c)}{g'(c)} + \frac{f'(c)}{g'(c)} - L \right| \\ &\leq \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)} \right| + \left| \frac{f(x) - f(y)}{g(x) - g(y)} - \frac{f'(c)}{g'(c)} \right| + \left| \frac{f'(c)}{g'(c)} - L \right|. \end{aligned} \tag{6}$$

We may bound the first term as  $y \rightarrow a^+$ , the second term is zero by construction, and the third term is bound by assumption as  $x \rightarrow a^+$  (and therefore  $c \rightarrow a^+$ ). So pick an  $\epsilon > 0$  and  $a < y < c < x$ . It follows that there are  $\delta_1, \delta_2 > 0$  so that

$$\begin{aligned} a < y < a + \delta_1 &\implies \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)} \right| < \frac{\epsilon}{2} \\ a < c < a + \delta_2 &\implies \left| \frac{f'(c)}{g'(c)} - L \right| < \frac{\epsilon}{2}. \end{aligned}$$

Clearly then, if we take  $a < x < \delta + a$  where  $\delta = \min\{\delta_1, \delta_2\}$ , we have from (6) that

$$\left| \frac{f(x)}{g(x)} - L \right| < \frac{\epsilon}{2} + (0) + \frac{\epsilon}{2} = \epsilon.$$

Notice that we may extend this argument to the left-handed limit, but the result holds even if only the right-handed limit exists. At any rate, the argument is identical, so we are done.

The only thing left to prove that we may extend this result to the case when  $a = \pm\infty$  and/or  $L = \pm\infty$ . This is actually easier than we might think! For instance, imagine  $a = -\infty$ . Since we have taken the right-handed limit in the above argument, we may extend to this case by applying the same argument to find  $-\infty < y < c < x$  satisfying the same conditions. Nothing else changes! (We may do the limit for  $a = \infty$  by considering  $x < c < y < \infty$ .)

We may similarly allow  $L = \pm\infty$  if we are careful to note that we will need  $|f'(c)/g'(c)| > M_1$  for arbitrary  $M_1 > 0$  to imply that  $|f(x)/g(x)| > M_2$  for arbitrary  $M_2$ . That is,  $f'(c)/g'(c) \rightarrow \infty (= L)$  implies  $f(x)/g(x) \rightarrow \infty (= L)$ . The details, for what they are worth, are left as an exercise.  $\square$